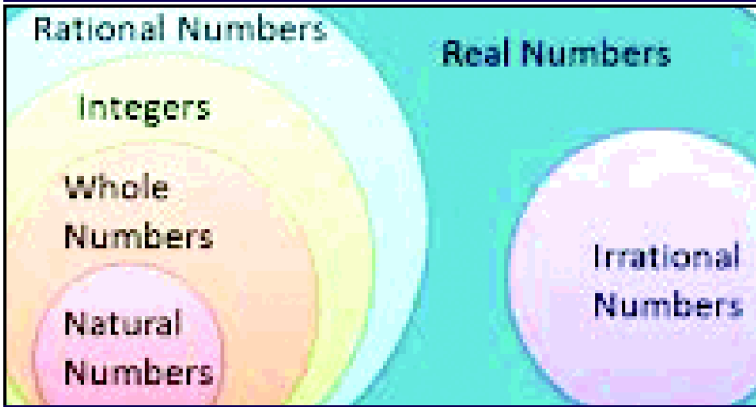
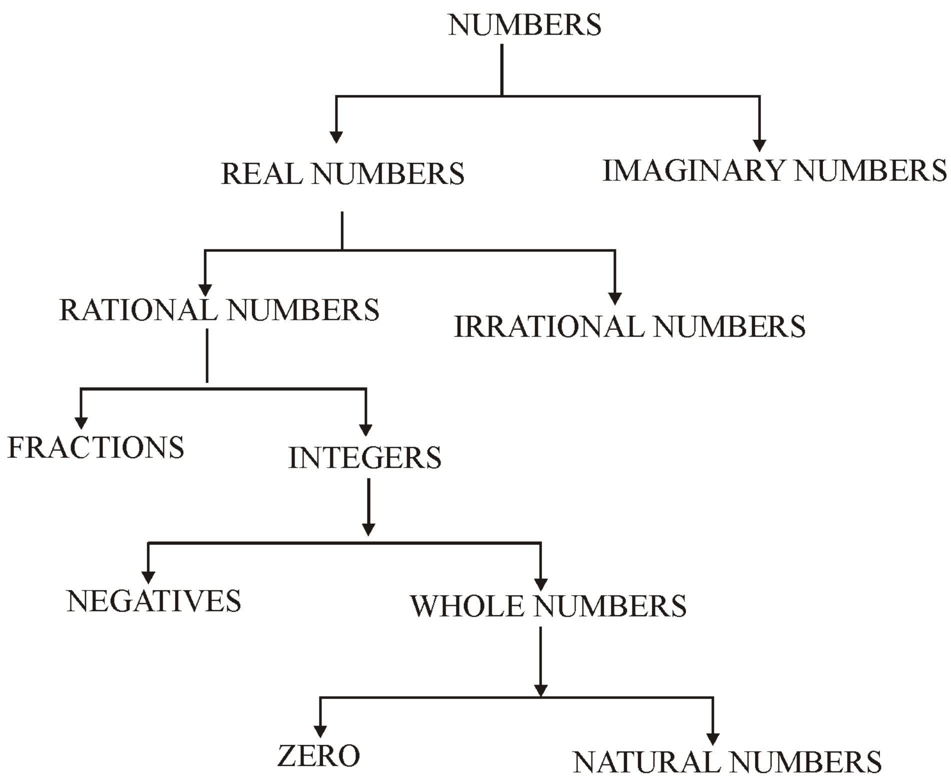
1 Number System



Topics Covered

- Classification of Numbers.
- Natural Numbers.
- Whole Numbers
- Integers
- Rational Numbers
- Irrational Numbers
- Real Numbers
- Surds
- Laws of Indices

CLASSIFICATION OF NUMBERS



Natural Numbers

All counting numbers are called natural numbers. If N is the set of natural numbers, then $N = \{1, 2, 3, 4, 5, \cdots \infty\}$

Important Notes

i) 1 is smallest natural number
ii) Adding 1 to each natural number, we get next natural number.

Whole Numbers

Natural numbers including zero represent the set of whole numbers. It is denoted by the symbol W.

$W = \{0, 1, 2, 3, 4, 5, \cdots \infty\}$

Important Notes

i) 0 is smallest whole number
ii) **Every natural number is whole number but every whole number is not natural number**

Integers

All natural numbers, (positive and negative) and 0, together form the set Z or I of all integers.

The set integers $Z = \{-\infty \cdots\cdots\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\cdots +\infty\}$

$Z = Z^{+} \cup \{0\} \cup Z^{-}$

$Z^{+} = \{1, 2, 3, 4, 5, 6, \cdots\cdots \infty\} = N$ is the set of all positive integers.

$Z^{-} = \{-1, -2, -3, -4, \cdots\cdots \infty\}$ is the set of all negative integers.

Important Notes

It has neither the greatest or least element

Rational Numbers

Any number that can be expressed in the form $\frac{p}{q}$ (where $q \neq 0$ and p, q are integers) is called a rational number.

Thus $Q = \left\{\frac{a}{b} \mid a, b \in z \text{ and } b \neq 0\right\}$

Example $\frac{4}{1}, \frac{5}{1}, \frac{0}{1}$ etc., $0.333\cdots = \frac{1}{3}$, $0.2 = \frac{2}{10} = \frac{1}{5}$

- All natural numbers, whole numbers and integers are rational numbers.
- Every terminating decimal is a rational number.
- Every recurring decimal (A non- terminating repeating decimal is called a recurring decimal.) is a rational number.

Important Notes

i) There exist infinite number of rational numbers between any two rational numbers. This property is known as the **density of rational numbers.**

PROPERTIES :

1. (i) The sum of two rational numbers is always rational. [Closure Property for addition]
   (ii)The product of two rational numbers is always rational. [Closure property for multiplication]

2. For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ we have

   (i) $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$ [commutative law of addition]

   (ii) $\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$ [commutative law of multiplication]

3. For any three rational numbers $\frac{a}{b},\frac{c}{d},\frac{e}{f}$ we have,

   (i) $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$ [Associatiive law of addition]

   (ii) $\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f}=\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)$ [Associatiive law of multiplication]

4. The difference of any two rational numbers is always rational.

5. If $\frac{a}{b}$ is a non-zero rational, then $\frac{b}{a}$ is called its reciprocal and $\frac{a}{b}\times\frac{b}{a}=1$

☞ **Finding rational numbers between two numbers** :

(A) Method I :

Find a rational number between a and b then, $\frac{a+b}{2}$ is a rational number lying between a and b.

E.g. Find a rational number between 2 and 7

Sol. Here $a=2,\ b=7$

then, a rational number between 2 and 7 $\frac{2+7}{2}=\frac{9}{2}$

(B) Method II :

Find n rational number between a and b **(when a and b is non fraction number)**then we use formula.

$$\frac{a(n+1)}{n+1},\frac{b(n+1)}{n+1}$$

E.g. Find 3 numbers between 4 and 5.

Sol. Here $a=4,\ b=5,\ n=3$

then, $\frac{a(n+1)}{n+1}=\frac{4(3+1)}{3+1}=\frac{16}{4}$. Again $\frac{b(n+1)}{n+1}=\frac{5(3+1)}{3+1}=\frac{20}{4}$

$\therefore\ \frac{16}{4}\left[\frac{17}{4},\frac{18}{4},\frac{19}{4}\right]\frac{20}{4}$

Hence rational numbers between 4 and 5 are $\frac{17}{4},\frac{18}{4},\frac{19}{4}$.

(C) Method III :

Find n rational number between a and b **(when a and b is fraction Number)** then we use formula

$$d = \frac{(b-a)}{n+1}$$

then n rational number lying between a and b are **($a + d$), ($a + 2d$), ($a + 3d$) .....($a + nd$)**

Remark : a = First Rational Number, b = Second Rational Number, n = No. of Rational Number.

E.g. Find 3 rational numbers between $\frac{7}{5}, \frac{9}{5}$.

Sol. ▶ Here $a = \frac{7}{5},\ b = \frac{9}{5},\ n = 3$

then $d = \left(\frac{b-a}{n+1}\right) = \frac{\frac{9}{5} - \frac{7}{5}}{3+1} = \frac{\frac{2}{5}}{4} = \frac{1}{10}$

$\therefore$ 3 rational numbers between $\frac{7}{5}$ and $\frac{9}{5}$ are $(a+d),\ (a+2d),\ (a+3d)$

then, 1st rational number $= (a+d) = \frac{7}{5} + \frac{1}{10} = \frac{15}{10}$

2nd rational number $= (a+2d) = \frac{7}{5} + \frac{2}{10} = \frac{16}{10}$

3rd rational number $= (a+3d) = \frac{7}{5} + \frac{3}{10} = \frac{17}{10}$

Hence 3 rational numbers between $\frac{7}{5}$ and $\frac{9}{5}$ are $\frac{14}{10}\left[\frac{15}{10}, \frac{16}{10}, \frac{17}{10}\right]\frac{18}{10}$

▶ Important Notes

NON-TERMINATING REPEATING DECIMAL NUMBERS

It has two types :

(a) Pure recurring decimals :

A decimal in which all the digit after the decimal point are repeated.

E.g. : $0.\overline{3},\ 0.\overline{16}, 0.\overline{123}$ are pure recurring decimals.

(b) Mixed recurring decimals :

A decimals in which at least one of the digits after the decimal point is not repeated and then some digit or digits are repeated.

E.g. $3.1\overline{6},\ 0.13\overline{5},\ 0.27\overline{85}$ are mixed recurring decimals.

Conversion recurring decimals to the form $\frac{p}{q}$

Method :

$$\left(\frac{p}{q}\right)\text{form} = \frac{\text{(Complete numbers)} - \text{(number formed by Nonrepeating digit)}}{\text{No. of 9 as no. of repeating digits after that write no. of 0 as no. of non repeating digits.}}$$

Ex. (i) $0.\overline{585} = \frac{585-0}{999} = \frac{195}{333} = \frac{65}{111}$

(ii) $0.12\overline{3} = \frac{123-12}{900} = \frac{111}{900} = \frac{37}{300}$

(iii) $25.6\overline{32} = 25 + \frac{632-6}{990} = 25 + \frac{626}{990} = \frac{25 \times 495 + 313}{495} = \frac{12688}{495}$

(iv) $15.7\overline{12} = 15\frac{712-7}{990} = 15\frac{705}{990} = 15\frac{141}{198} = 15\frac{47}{66} = \frac{15 \times 66 + 47}{66} = \frac{1037}{66}$

Irrational Numbers

A number is called irrational number, if it can not be written in the form $\frac{p}{q}$, where p & q are integers and $q \neq 0$. All Non-terminating & Non-repeating decimal numbers are Irrational numbers.

Ex. $\sqrt{2}, \sqrt{5}, 3\sqrt{2}, 1+\sqrt{3}, \sqrt{3+\sqrt{2}}$, π, etc...

Important Notes

i) The sum and difference of a rational and an irrational number is irrational;
E.g. : 3 is rational and $\sqrt{2}$ is irrational and so $3+\sqrt{2}$, $3-\sqrt{2}$ are irrational.

ii) The sum of two irrationals may be rational or irrational.
E.g. : $\sqrt{3}+1$ and $1-\sqrt{3}$ both are irrational but the sum is rational $\left(\sqrt{3}+1+1-\sqrt{3}=2\right)$

iii) Product of an irrational with an irrational is not always irrational
E.g. $\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$ a rational number.
$\left(1+\sqrt{3}\right) \times \sqrt{2} = \sqrt{2} + \sqrt{6}$ a irrational number.

iv) Product of non-zero rational number with an irrational number is always irrational number.
$\frac{1}{2} \times \sqrt{5} = \frac{\sqrt{5}}{2}$ a irrational number.

Concept Illustrator

1. Prove that $\sqrt{2}$ is not rational.

Sol. Since $1 < 2 < 4$, hence, $\sqrt{1} < \sqrt{2} < \sqrt{4}$ or $1 < \sqrt{2} < 2$, which shows that $\sqrt{2}$ cannot be an integer.

If possible, let us assume, $\sqrt{2}$ is rational. Then we can write, $\sqrt{2} = \frac{p}{q}$ ...(i) where p and q are positive integers prime to each other *i.e.*, they have no common factor other than 1 and $q > 1$ (since $\sqrt{2}$ is not an integer)

Now, $\frac{p}{q}=\sqrt{2}$ or, $\frac{p^2}{q^2}=2$ (squaring) or, $\frac{p^2}{2q}=q$ or, $\frac{p^2}{q}=2q$

By hypothesis, p and q are positive integers prime to each other. Again $q>1$. Therefore, $\frac{p^2}{q}$ represents a positive rational number which is not an integer. But $2q$ represents a positive integer. Therefore, from (1) we get, a positive rational number which is not an integer = a positive integer. Clearly, it is impossible. Hence, our assumption cannot be true *i.e.*, $\sqrt{2}\neq\frac{p}{q}$. In other words, $\sqrt{2}$ is not a rational number (proved)

2. Show that $\sqrt[3]{4}$ is not a rational number.

Sol. We have $1<4<8$

Whence $\sqrt[3]{1}<\sqrt[3]{4}<\sqrt[3]{8}$, *i.e.*, $1<\sqrt[3]{4}<2$

which shows that $\sqrt[3]{4}$ is not an integer.

If possible, let $\sqrt[3]{4}=\frac{p}{q}$ ...(i), where p, q are mutually prime integers and $q>1$.

Now cubing both sides of (1) we have

$\frac{p^3}{q^3}=4$ or, $4q^2=\frac{p^3}{q}$.

Since p, q are prime to each other, so p^3 and q are also prime to each other, *i.e.*, there is no common factor between p^3 and q. In other words q does not divide p^3. So $\frac{p^3}{q}$ is not an integer. But $4q^2$ is clearly an integer whence we see that equality (1) does not hold [since L.H.S. of (1) is an integer while R.H.S. of (1) is not an integer]. Consequently, our hypothesis that $\sqrt[3]{4}$ is rational must be wrong. So $\sqrt[3]{4}$ is not a rational number.

3. Find rational value of p so that $\sqrt{p^2+p+1}$

Suppose x and p are both rational, so that $(x-p)$ is rational.

Now, $x-p=\sqrt{p^2+p+1}-p$

Let $x-p=y$, where y is rational.

So, from (1) we have

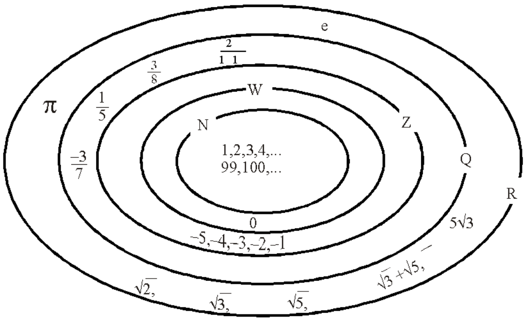
$$y=\sqrt{p^2+p+1}-p\Rightarrow y+p=\sqrt{p^2+p+1}\Rightarrow y^2+2py+p^2=p^2+p+1$$

$$\Rightarrow (2y-1)p=1-y^2\Rightarrow p=\frac{1-y^2}{2y-1}=\frac{y^2-1}{1-2y}$$

So, for $y\neq\frac{1}{2}$, the given expression is rational for all rational values of p, where $p=\frac{y^2-1}{1-2y}$

Real Numbers

The collection of rational numbers and irrational numbers is called the set of real numbers. If Q is the set of rational numbers and P is the set of irrational numbers then $Q\cap P=\phi$ and every real number is either rational or irrational.



N : The set of natural numbers,

W : The set of whole numbers,

Z : The set of Integers,

Q : The set of rationals,

R : The set of Real Numbers.

Important Notes

i) Two real numbers a and b, either $a = b$, $a > b$, or $a < b$

ii) The real numbers obey all the laws of algebra that the rational numbers obey.

iii) The sum, difference and product of two real numbers is real.

iv) The division of a real number by non-zero real number is real.

v) Every real number has a negative real number. 0 is its own negative number.

vi) The sum, difference, product quotient of a rational number and an irrational number is irrational.

Thus, $2+\sqrt{3}$, $-3+\sqrt{5}$, $\dfrac{1}{4+\sqrt{2}}$, etc. are irrational numbers.

vii) The sum, difference, product and quotient of two irrational numbers need not be irrational.

viii) Between two unequal real numbers there are innumerable real numbers. Many of these are rational and many are irrational

☞ To find a real number between two unequal real numbers :

i) If a, b are two real numbers, $\dfrac{a+b}{2}$ is a real number lying between a and b.

ii) If a, b are two positive real numbers, $\sqrt{ab}$ is an irrational number lying between a and b.

iii) If a and b are two positive real numbers such that $a \times b$ is not a perfect square of a rational number, $\sqrt{ab}$ is an irrational number lying between a and b.

Example $\sqrt{2\times 5}$, *i.e.*, $\sqrt{10}$ is an irrational number between 2 and 5 because 2×5 *i.e.*, 10 is not a perfect square of a rational number.

Absulate value of a real number :

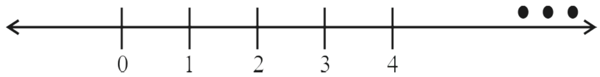
The absulate value (or modulus) of a real number a is denoted by $|a|$, and is defined as

$$|a| = \begin{cases} a, & if \;\; a \ge 0 \\ -a, & if \;\; a < 0 \end{cases}$$

◆ Representation of numbers of the number line :

Real Numbers :

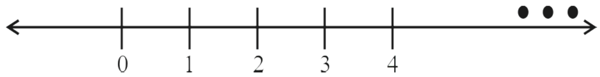
Draw a line, Mark a point on it which represents 0 (zero). Now on the right hand side of zero (0), mark points at equal intervals of length, as shown below :



These points represent natural numbers 1, 2, 3, ⋯ respectively. The three dots on number line indicate the continuation of these numbers indefinitely.

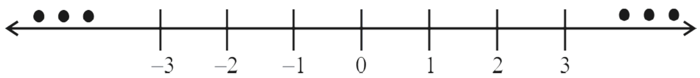
Whole Numbers :

This is similar as above, but with the inclusion of 0 in the number line it is as follows :



Integers :

Draw a line, Mark a point on it which represents 0 (zero).



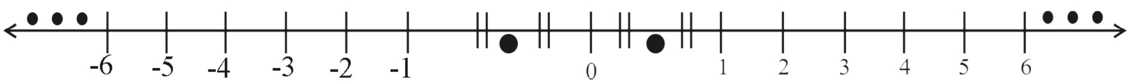
Three dots on either side show the continuation of integers indefinitely on each side.

Rational Numbers :

Rational numbers can be represented by some points on the number line.

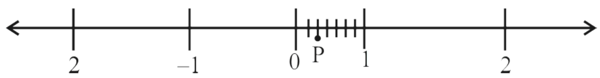
Draw a line. Mark a point on it which represents 0 (zero)

Set equal distances on both sides of 0. Each point on the division represents an integer as shown below.



The length between two successive integers is called a unit length.

Let us consider a rational number $\frac{2}{7}$



Divide unit length between 0 and 1 into 7 equal parts; call them sub-divisions.

The point at the line indicating the second sub-division represents $\frac{2}{7}$.

In this way any rational number can be represented on the number line.

◆ Representation of Irrational numbers on the number line :

We use the Pythagorus property of a right angled trangle, according to which, in a right angled tringle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Consider the number line ℓ and a perpendicular line ℓ_1 to it.

Let $OA = 1$ unit and $OP = 1$ unit

Let $OAXP$ be a square.

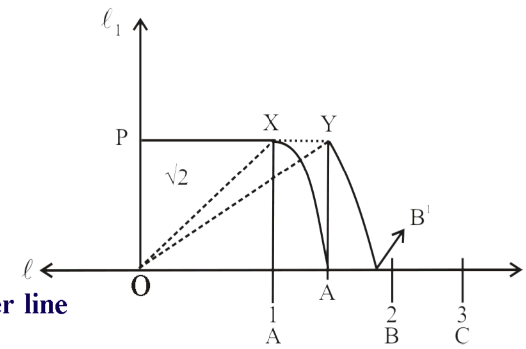
$OX = \sqrt{1^2 + 1^2} = \sqrt{2}$

Taking O as the centre and OX as the radius

cut the number

line at the point A^1

$\Rightarrow$ i.e., $OX = OA^1 = \sqrt{2}$



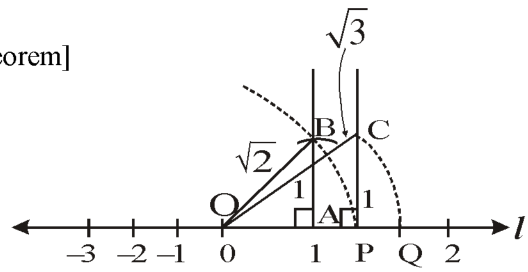
Concept Illustrator

- **Representation of Irrational numbers on a number line**

To represent $\sqrt{3}$ on the real number line :

OC= $\sqrt{OP^2 + PC^2}$ [By Pythagorus theorem]

$= \sqrt{(\sqrt{2})^2 + (1)^2} = \sqrt{2+1} = \sqrt{3}$



Then, OC = OQ = $\sqrt{3}$ unit

Thus, the point Q represent $\sqrt{3}$ on the number line.

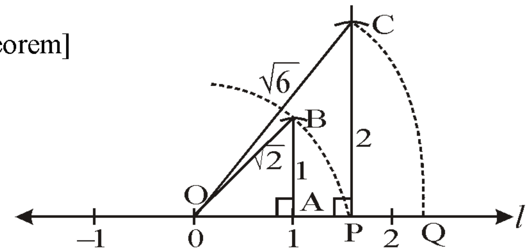
To represent $\sqrt{6}$ on the real number line :

OC = $\sqrt{OP^2 + PC^2}$ [By Pythagorus theorem]

$= \sqrt{(\sqrt{2})^2 + (2)^2} = \sqrt{2+4} = \sqrt{6}$

Then, OC = OQ = $\sqrt{6}$ unit

Thus, the point Q represent $\sqrt{6}$ on the number line.

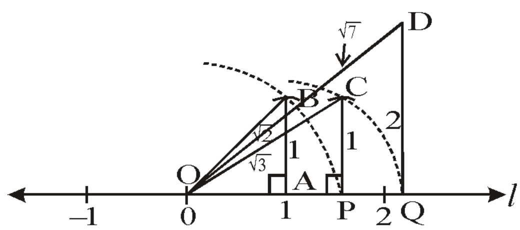


To represent $\sqrt{7}$ on the real number line :

OD = $\sqrt{OQ^2 + DQ^2} = \sqrt{(\sqrt{3})^2 + (2)^2} = \sqrt{3+4} = \sqrt{7}$

Then, OD = OQ = $\sqrt{7}$ unit

Thus, the point Q represent $\sqrt{7}$ on the number line.

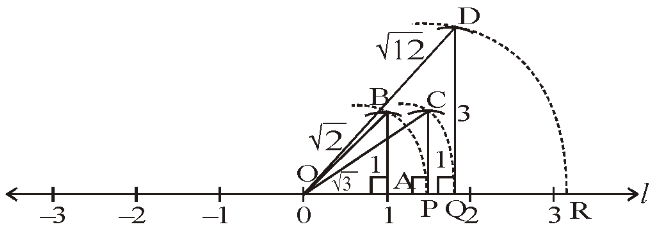


To represent $\sqrt{12}$ on the real number line :

OD = $\sqrt{OQ^2 + DQ^2}$

$= \sqrt{(\sqrt{3})^2 + (3)^2}$

$= \sqrt{3+9} = \sqrt{12}$



Then, OD = OR = $\sqrt{12}$ unit

Thus, the point R represent $\sqrt{12}$ on the number line.

Surds

Algebraic irrational numbers are said to be surds. Thus surds are irrational roots of equations with rational coefficients.

Example : $\sqrt{2}$ is a surd, for $\sqrt{2}$ is an irrational root of the equatoin $x^2 - 2 = 0$, which is an equation with rational coefficients, $3^{\frac{1}{3}}$ is surd, for it is an irrational root of the equation $x^3 - 3 = 0$; $3+\sqrt{2}$ is a surd, for it is the root of the equation $x^2 - 6x + 7 = 0$ which is with rational coefficients.

- $\sqrt{3}$ and $2^{\frac{1}{2}}$ are surds of the second order or quadratic surds.
- $\sqrt[3]{4}$, $5^{\frac{2}{3}}$, $x^{\frac{1}{3}}$ are surds of the third order or cubic surds.

Similarly, $\sqrt[4]{7}$, $\sqrt[n]{Q}$ are surds of the 4th and nth orders respectively.

Important Notes

i) π, e etc. are not surds. $\sqrt{4}, \sqrt[3]{27}, \sqrt[4]{\frac{16}{81}}$ are not surds. for $\sqrt{4} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[4]{\frac{16}{81}} = \frac{2}{3}$.

All are rational numbers, but surds are irrational numbers.

ii) **Different surds :**

a) **Simple pure surd :** A surd with only one term and without any rational co-efficient is known as a simple pure surd. For example, $\sqrt{3}$, $\sqrt[3]{5}$ etc.

b) **Simple mixed surd :** A surd with only one term and with a rational coefficient is known as a simple mixed surd. For example. $5\sqrt[3]{7}$, $3\sqrt{5}$, etc.

c) **Compound surd :** If surds more than one are connected by '+' or '–' sign, then that quantity is known as a compound surd. For example $\sqrt{3}+\sqrt[3]{5}, +\sqrt{5}+\sqrt[4]{5}$, etc.

d) **Binomial and Trinomial surds :** The algebraic sum of two surds or a rational quantity and *a* surd is said to be a binomial surd $2\sqrt{3}+\sqrt{5}$, $\sqrt{3}+2\sqrt{2}$, $\sqrt{7}-\sqrt{3}$. Thus are binomial surds. Similarly, $\sqrt{5}+\sqrt{3}+\sqrt{2}$, $3+\sqrt{7}-\sqrt{2}$ are trinomial surds.

e) **Conjugate surd and Complementary surd :** In two binomial surds, if the two terms are identical but signs between the surds of opposite nature, one surd is known as conjugate or complementary to the other surd.

Example $3+\sqrt{5}$ and $3-\sqrt{5}$ or $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$

Hence, the product of two conjugate surds is always rational and evidently each is the rationalising factor of the other.

iii) Comparison of Surds :

a) For equiradical surds $\sqrt[3]{18} > \sqrt[3]{15}$ since 18>15.

b) For comparison between two surds of different orders we express them to surds of the same order. Thus, to compare between $\sqrt[3]{4}$ and $\sqrt[5]{6}$ we express them to surds of the same order as follows :

Clearly, the orders of the given surds are 3 and 5 respectively and the L.C.M. of 3 and 5 is 15.

Therefore, $\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{5}{15}} = \sqrt[15]{4^5} = \sqrt[15]{1024}$ and $\sqrt[5]{6} = 6^{\frac{1}{5}} = 6^{\frac{3}{15}} = \sqrt[15]{6^3} = \sqrt[15]{216}$

Now, 1024 > 216; therefore, $\sqrt[15]{1024} > \sqrt[15]{216}$ *i.e.,* $\sqrt[3]{4} > \sqrt[5]{6}$

iv) Addition and Subtraction of surds :

To find the sum (or difference) of two or more surds—

a) express each surd in its simplest mixed form;

b) then find the sum (or difference) of rational co-efficients of surds.

c) finally, to get the required sum (or difference) of like surds multiply the result obtained in (b) by the surd-factor of like surds.

The sum (or difference) of unlike surds is expressed in a number of terms by connecting them with plus (+) or minus (–) sign.

E.g. $\sqrt{32} - 2\sqrt{18} + 5\sqrt{2} + 2^{\frac{3}{2}}$

Sol. $\sqrt{32} - 2\sqrt{18} + 5\sqrt{2} + 2^{\frac{3}{2}} = \sqrt{16 \times 2} - 2\sqrt{9 \times 2} + 5\sqrt{2} + \sqrt{2^3} = 4\sqrt{2} - 6\sqrt{2} + 5\sqrt{2} + 2\sqrt{2}$

$= (4 + 5 + 2)\sqrt{2} - 6\sqrt{2} = 11\sqrt{2} - 6\sqrt{2} = (11 - 6)\sqrt{2} = 5\sqrt{2}$

v) Multiplication and division of surds :

The surds of the same order can be multiplied according to the law $\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy}$

Note :

When the surds to be multiplied or divided are not of the same order, they have to be necessarily brought to the same order before the operation is done.

E.g. $\sqrt{8} \times 3\sqrt{3} \times 2\sqrt{45} = \sqrt{2^2 \times 2} \times 3\sqrt{3} \times 2\sqrt{3^2 \cdot 5} = 2\sqrt{2} \times 3\sqrt{3} \times 2 \times 3\sqrt{5}$

$= (2 \times 3 \times 6) \times (\sqrt{2} \times \sqrt{3} \times \sqrt{5}) = 36 \times \sqrt{2 \times 3 \times 5} = 36\sqrt{30}$

E.g. $(\sqrt{x} + 1) \div (\sqrt{y} + 1) = \dfrac{\sqrt{x} + 1}{\sqrt{y} + 1} = \dfrac{(\sqrt{x} + 1)(\sqrt{y} - 1)}{(\sqrt{y} + 1)(\sqrt{y} - 1)} = \dfrac{\sqrt{x}\sqrt{y} + \sqrt{y} - \sqrt{x} - 1}{(\sqrt{y})^2 - 1} = \dfrac{\sqrt{xy} + \sqrt{y} - \sqrt{x} - 1}{y - 1}$

vi) The method of converting a given surd into a rational number on multiplication by another suitable surd is called **rationalisation of surds.** In this case the multiplying surd is called the raionalising factor of the given surd and conversely.

- $\sqrt{a} + \sqrt{b}$ is rationalizing factor of $\sqrt{a} - \sqrt{b}$ and vice versa
- $\sqrt[3]{a} + \sqrt[3]{b}$ is rationalizing factor of $\sqrt[3]{a} - \sqrt[3]{b}$
- $\sqrt[3]{a} - \sqrt[3]{b}$ is rationalizing factor of $\sqrt[3]{a} + \sqrt[3]{b}$

- Rationalizing factor of $2^{\frac{1}{3}}+2^{\frac{-1}{3}}$ is $2^{\frac{2}{3}}-1+2^{\frac{1}{3}}$
- Rationalizing factor of $2^{\frac{1}{3}}-2^{\frac{-1}{3}}$ is $2^{\frac{2}{3}}+1+2^{\frac{1}{3}}$

vii) If $a+\sqrt{b}=x+\sqrt{y}$ and a and x are both rationals and $\sqrt{b}$ and $\sqrt{y}$ are both surds then $a=x$ and $b=y$.

viii) If $a-\sqrt{b}=x-\sqrt{y}$, then $a=x$ and $b=y$

ix) If $a+\sqrt{b}=0\left(\text{or, } a-\sqrt{b}=0\right)$ then $a=0$ and $b=0$

☞ **Square Root of Quadratic Surds :**

(a) $\left(\sqrt{x}+\sqrt{y}\right)^2=\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+2\sqrt{x}\cdot\sqrt{y}=(x+y)+2\sqrt{xy}$

$=a+\sqrt{b},$ where $a=(x+y)$ and $\sqrt{b}=2\sqrt{xy}$

$\therefore\ \sqrt{a+\sqrt{b}}=\pm\left(\sqrt{x}+\sqrt{y}\right)$

Thus the square root of $\left(a+\sqrt{b}\right)$ is either $+\left(\sqrt{x}+\sqrt{y}\right)$ or $-\left(\sqrt{x}+\sqrt{y}\right)$

E.g. Find the square roots of $\frac{1}{2}\left(2+\sqrt{3}\right)$

Sol. $\frac{1}{2}\left(2+\sqrt{3}\right)=\frac{1}{4}\left(4+2\sqrt{3}\right)=\frac{1}{4}\left[\left(\sqrt{3}\right)^2+1^2+2\cdot\sqrt{3}\cdot 1\right]=\left[\frac{1}{2}\left(\sqrt{3}+1\right)\right]^2$

$\therefore$ Square roots are $\pm\frac{1}{2}\left(\sqrt{3}+1\right)$

(b) $\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2=x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}=a+\sqrt{b}+\sqrt{c}+\sqrt{d}$

where, $a=x+y+z,\ \sqrt{b}=2\sqrt{xy},\ \sqrt{c}=2\sqrt{yz},\ \sqrt{d}=2\sqrt{zx}$

$\therefore\ \sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\pm\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$

Thus the square root of $\left(a+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)$ is either $+\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$ or $-\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$

E.g. Find the square roots of $8+2\sqrt{2}-2\sqrt{5}-2\sqrt{10}$

Sol. Given exp. $=2+1+5+2\cdot\sqrt{2}\cdot 1+2\cdot\left(-\sqrt{5}\right)\cdot 1+2\cdot\sqrt{2}\cdot\left(-\sqrt{5}\right)=\left(\sqrt{2}+1-\sqrt{5}\right)^2$

$\therefore$ Required square roots are $\pm\left(\sqrt{2}+1-\sqrt{5}\right)$

Concept Illustrator

1. Simplify $\frac{\sqrt{2}\left(2+\sqrt{3}\right)}{\sqrt{3}\left(\sqrt{3}+1\right)}-\frac{\sqrt{2}\left(2-\sqrt{3}\right)}{\sqrt{3}\left(\sqrt{3}-1\right)}$

Sol. Thve given exp. $\dfrac{\sqrt{2}\left(2+\sqrt{3}\right)\times\sqrt{3}\left(\sqrt{3}-1\right)}{\sqrt{3}\left(\sqrt{3}+1\right)\times\sqrt{3}\left(\sqrt{3}-1\right)}-\dfrac{\sqrt{2}\left(2-\sqrt{3}\right)\times\sqrt{3}\left(\sqrt{3}+1\right)}{\sqrt{3}\left(\sqrt{3}-1\right)\times\sqrt{3}\left(\sqrt{3}+1\right)}$

$$=\frac{3\sqrt{2}+\sqrt{6}}{3(3-1)}-\frac{3\sqrt{2}-\sqrt{6}}{3(3-1)}=\frac{3\sqrt{2}+\sqrt{6}-3\sqrt{2}+\sqrt{6}}{6}=\frac{2\sqrt{6}}{6}=\frac{\sqrt{6}}{3}$$

2. Rationalise the denominator of the fraction $\dfrac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}$

Sol. The given fraction $=\dfrac{\left(\sqrt{a+1}+\sqrt{a-1}\right)^2}{\left(\sqrt{a+1}-\sqrt{a-1}\right)\left(\sqrt{a+1}+\sqrt{a-1}\right)}$

$$=\frac{a+1+a-1+2\sqrt{a^2-1}}{(a+1)-(a-1)}=\frac{2a+2\sqrt{a^2-1}}{2}=a+\sqrt{a^2-1}$$

3. If $x=\dfrac{\sqrt{3}}{2}$, find the value of $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Sol. The given exp. $=\dfrac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{1+x}-\sqrt{1-x}\right)}$

$$=\frac{1+x+1-x-2\sqrt{1-x^2}}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}=\frac{2-2\sqrt{1-x^2}}{1+x-(1-x)}=\frac{2\left(1-\sqrt{1-x^2}\right)}{2x}$$

$$=\frac{1-\sqrt{1-x^2}}{x}=\frac{1-\sqrt{1-\frac{3}{4}}}{\frac{\sqrt{3}}{2}}=\frac{1-\sqrt{\frac{1}{4}}}{\frac{\sqrt{3}}{2}}=\frac{1-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}=\frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{\sqrt{3}}{3}$$

4. Show that $\sqrt{6}+\sqrt{2}<\sqrt{5}+\sqrt{3}$

Sol. $\left(\sqrt{6}+\sqrt{2}\right)^2=6+2+2\sqrt{12}=8+2\sqrt{12}$ and $\left(\sqrt{5}+\sqrt{3}\right)^2=5+3+2\sqrt{15}=8+2\sqrt{15}$

$\because \sqrt{12}<\sqrt{15}$ $\quad\therefore 8+2\sqrt{12}<8+2\sqrt{15}$

$\therefore \sqrt{6}+\sqrt{2}<\sqrt{5}+\sqrt{3}$

5. If $x=\frac{\sqrt{5}-1}{\sqrt{5}+1}$ and $y=\frac{\sqrt{5}+1}{\sqrt{5}-1}$, find the value of $\frac{x^2}{y}+\frac{y^2}{x}$

Sol. $x+y=\frac{\sqrt{5}-1}{\sqrt{5}+1}+\frac{\sqrt{5}+1}{\sqrt{5}-1}=\frac{(\sqrt{5}-1)^2+(\sqrt{5}+1)^2}{(\sqrt{5}+1)(\sqrt{5}-1)}=\frac{2(5+1)}{5-1}=\frac{12}{4}=3$

and $xy=\frac{\sqrt{5}-1}{\sqrt{5}+1}\times\frac{\sqrt{5}+1}{\sqrt{5}-1}=1$

Given expression $=\frac{x^2}{y}+\frac{y^2}{x}=\frac{x^3+y^3}{xy}$

$=\frac{(x+y)^3-3xy(x+y)}{xy}=\frac{(3)^3-3\cdot1\cdot3}{1}=\frac{27-9}{1}=18$

6. If $x=2+\sqrt{3}$ show that $x^3-2x^2-7x+2=0$

Sol. $\because\ x=2+\sqrt{3}$ $\qquad\therefore\ x-2=\sqrt{3}$

or, $(x-2)^2=(\sqrt{3})^2$ $\qquad$ or, $x^2-4x+4=3$

or, $x^2-4x+1=0$

L.H.S. $=x^3-2x^2-7x+2=x^3-4x^2+x+2x^2-8x+2$

$=x(x^2-4x+1)+2(x^2-4x+1)=(x^2-4x+1)(x+2)=0\times(x+2)=0$ R.H.S. (Proved)

7. If $x=1+\sqrt{2}$ and $ma=1+\sqrt{1-a^2}$ then show that $\frac{a}{2m}(1+mx)\left(1+\frac{m}{x}\right)=1+\sqrt{2}a$

Sol. $x=\sqrt{2}+1,\ \frac{1}{x}=\sqrt{2}-1\Rightarrow x+\frac{1}{x}=2\sqrt{2}$ and $m=\frac{1+\sqrt{1-a^2}}{a}$

$\frac{1}{m}=\frac{1-\sqrt{1-a^2}}{a}\Rightarrow m+\frac{1}{m}=\frac{2}{a}$. Now $\frac{a}{2m}(1+mx)\left(1+\frac{m}{x}\right)=\frac{a}{2m}\left\{1+m\left(x+\frac{1}{x}\right)+m^2\right\}$

$=\frac{a}{2}\left(\frac{1}{m}+m\right)+\frac{a}{2}\left(x+\frac{1}{x}\right)=1+\sqrt{2}a$

8. Solve $\sqrt{x}\left(9^{\sqrt{x^2-3}}-3^{\sqrt{x^2-3}}\right)=3^{2\sqrt{x^2-3}+1}-3^{\sqrt{x^2-3}+1}+6\sqrt{x}-18$

Sol. $\sqrt{x}\left(9^{\sqrt{x^2-3}}-3^{\sqrt{x^2-3}}\right)=3^{2\sqrt{x^2-3}+1}-3^{\sqrt{x^2-3}+1}+6\sqrt{x}-18$

or $\sqrt{x}\left(3^{2\sqrt{x^2-3}}-3^{\sqrt{x^2-3}}\right)=3.3^{2\sqrt{x^2-3}}-3.3^{\sqrt{x^2-3}}+6\sqrt{x}-18$

or $3^{2\sqrt{x^2-3}}\cdot(\sqrt{x}-3)-3^{\sqrt{x^2-3}}(\sqrt{x}-3)-6(\sqrt{x}-3)=0$

or $(\sqrt{x}-3)\left(3^{2\sqrt{x^2-3}}-3^{\sqrt{x^2-3}}-6\right)=0$

$\therefore\ \sqrt{x}-3=0,$ or $x=9$ or $3^{2\sqrt{x^2-3}}-3^{\sqrt{x^2-3}}-6=0$

or $\left(3^{\sqrt{x^2-3}}-3\right)\left(3^{\sqrt{x^2-3}}+2\right)=0$ $\therefore\ 3^{\sqrt{x^2-3}}-3=0$

or $3^{\sqrt{x^2-3}}=3$ or, $\sqrt{x^2-3}=1$ or $x^2-3=1$

or $x^2=4$ $\therefore\ x=2$ (Taking +ve value only)

Again $3^{\sqrt{x^2-3}}=-2$. (This gives the value of x as imaginary and hence rejected)

$\therefore$ The required solution is $x=9$ and $x=2$

9. Solve for x: $\left(5+2\sqrt{6}\right)^{x^2-5}+\left(5-2\sqrt{6}\right)^{x^2-5}=10$

Sol. $\left(5+2\sqrt{6}\right)^{x^2-5}+\left(5-2\sqrt{6}\right)^{x^2-5}=10$

or $\left(5+2\sqrt{6}\right)^{x^2-5}+\left[\dfrac{\left(5+2\sqrt{6}\right)\left(5-2\sqrt{6}\right)}{5+2\sqrt{6}}\right]^{x^2-5}=10$ or $\left(5+2\sqrt{6}\right)^{x^2-5}+\left[\dfrac{1}{5+2\sqrt{6}}\right]^{x^2-5}=10$

or $\left(5+2\sqrt{6}\right)^{x^2-5}+\dfrac{1}{\left(5+2\sqrt{6}\right)^{x^2-5}}=10$ $\cdots(1)$ Putting, $\left(5+2\sqrt{6}\right)^{x^2-5}=a$ $\cdots(2)$

we have $a+\dfrac{1}{a}=10$, or $a^2-10a+1=0$

$$\therefore\ a=\frac{-(10)\pm\sqrt{(-10)^2-4.1.1}}{2.1}=\frac{10\pm\sqrt{96}}{2}=\frac{10\pm4\sqrt{6}}{2}$$

$=5\pm2\sqrt{6}=\left(5+2\sqrt{6}\right),\ \left(5-2\sqrt{6}\right)$

Putting $a=5+2\sqrt{6}$ in (2) we have, $\left(5+2\sqrt{6}\right)^{x^2-5}=5+2\sqrt{6}$

or $x^2-5=1$ or $x^2=6,$ or $x=\pm\sqrt{6}$

Again, putting $a=5-2\sqrt{6}$ in (2), we get

$$\left(5+2\sqrt{6}\right)^{x^2-5}=5-2\sqrt{6}=\frac{1}{5+2\sqrt{6}}=\left(5+2\sqrt{6}\right)^{-1}$$

$\therefore\ x^2-5=-1$ or $x^2=4$ or $x=\pm2$

10. Show that $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}+\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\sqrt{-\sqrt{3}+\sqrt{3+8\sqrt{7+4\sqrt{3}}}}$ is a rational number. Find its simplest value.

Sol. $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}+\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\sqrt{-\sqrt{3}+\sqrt{3+8\sqrt{7+4\sqrt{3}}}}$

$$=\frac{\sqrt{6}\left(\sqrt{3}-\sqrt{2}\right)}{1}+\frac{3\sqrt{2}\left(\sqrt{6}-\sqrt{3}\right)}{6-3}-\frac{4\sqrt{3}\left(\sqrt{6}-\sqrt{2}\right)}{6-2}+\sqrt{-\sqrt{3}+\sqrt{3+8\sqrt{\left(2+\sqrt{3}\right)^2}}}$$

$$=\left(3\sqrt{2}-2\sqrt{3}\right)+\left(2\sqrt{3}-\sqrt{6}\right)-\left(3\sqrt{2}-\sqrt{6}\right)+\sqrt{-\sqrt{3}+\sqrt{3+16+8\sqrt{3}}}$$

$=\sqrt{-\sqrt{3}+\sqrt{\left(4+\sqrt{3}\right)^2}}=\sqrt{-\sqrt{3}+4+\sqrt{3}}=2$, which is a rational number.

Its simplest value is 2.

11. If $x=\frac{4\sqrt{15}}{\sqrt{5}+\sqrt{3}}$ show that $\frac{x+\sqrt{20}}{x-\sqrt{20}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}=2$

$\because$ $x=\frac{2\cdot 2\cdot\sqrt{3}\cdot\sqrt{5}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{20}\cdot 2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ or, $\frac{x}{\sqrt{20}}=\frac{2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

Now, comp.-div. we have $\frac{x+\sqrt{20}}{x-\sqrt{20}}=\frac{\sqrt{5}+3\sqrt{3}}{\sqrt{3}-\sqrt{5}}$ $\cdots(1)$

Again $\frac{x}{\sqrt{12}}=\frac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$, By comp.-div. we have $\frac{x+\sqrt{12}}{x-\sqrt{12}}=\frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ $\cdots(2)$

Adding equation (1) and (2) we get,

$$\frac{x+\sqrt{20}}{x-\sqrt{20}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}=\frac{\sqrt{5}+3\sqrt{3}}{\sqrt{3}-\sqrt{5}}-\frac{3\sqrt{5}+\sqrt{3}}{\sqrt{3}-\sqrt{5}}=\frac{2\sqrt{3}-2\sqrt{5}}{\sqrt{3}-\sqrt{5}}=2$$

12. Find the square root $a+b+\sqrt{2ab+b^2}$

Sol. Suppose, $\sqrt{(a+b)+\sqrt{2ab+b^2}}=\sqrt{x}+\sqrt{y}$

Squaring we have, $a+b+\sqrt{2ab+b^2}=x+y+2\sqrt{xy}$

$\therefore$ $x+y=a+b$ $\cdots(1)$ and $2\sqrt{xy}=\sqrt{2ab+b^2}$

$\therefore$ $(x+y)^2=(a+b)^2$ and $4xy=2ab+b^2$

Now, $(x-y)^2=(x+y)^2-4xy=(a+b)^2-2ab-b^2=a^2$

$\therefore$ $x-y=\pm a$ $\cdots(2)$ Solving (1) and (2) we have

$x=a+\frac{b}{2}=\frac{2a+b}{2}$ and $y=\frac{b}{2}$ or, $x=\frac{b}{2}$ and $y=\frac{2a+b}{2}$

$\therefore$ The required square root $=\pm\left(\sqrt{\frac{2a+b}{2}}+\sqrt{\frac{b}{2}}\right)$

13. Find the square root of $\dfrac{6+2\sqrt{3}}{33-19\sqrt{3}}$

Sol. $33-19\sqrt{3}=\sqrt{3}\left(11\sqrt{3}-19\right)$

Now $\dfrac{\left(6+2\sqrt{3}\right)\left(11\sqrt{3}+19\right)}{\sqrt{3}\left(11\sqrt{3}-19\right)\left(11\sqrt{3}+19\right)}=\dfrac{66\sqrt{3}+114+66+38\sqrt{3}}{\sqrt{3}\left(363-361\right)}$

$=\dfrac{180+104\sqrt{3}}{\sqrt{3}\times 2}=52+30\sqrt{3}=25+27+2\cdot5\cdot3\sqrt{3}=\left(5+3\sqrt{3}\right)^2$

$\therefore$ Required square root $=\pm\left(5+3\sqrt{3}\right)$

14. Find the value of $\left(2x^4-8x^3-5x^2+26x-28\right)$ when $x=1+\sqrt{2}+\sqrt{3}$

Sol. $(x-1)^2=\left(\sqrt{3}+\sqrt{2}\right)^2$ or, $x^2-2x+1=5+2\sqrt{6}$ or $x^2-2x-4=2\sqrt{6}$ $\cdots$ (1)

Again by squaring we get, $x^4+4x^2+16-4x^3-8x^2+16x=24$

or $2x^4-8x^3-8x^2+32x=16$

or $2x^4-8x^3-5x^2+26x-28=3x^2-6x-12=3\left(x^2-2x-4\right)=3\times2\sqrt{6}=6\sqrt{6}$

15. Find the value of x $(a+x)^{\frac{2}{3}}+2(a-x)^{\frac{2}{3}}=3\left(a^2-x^2\right)^{\frac{1}{3}}$

Sol. $(a+x)^{\frac{2}{3}}+2(a-x)^{\frac{2}{3}}=3(a+x)^{\frac{1}{3}}(a-x)^{\frac{1}{3}}$

Let $(a+x)^{\frac{1}{3}}=p$ and $(a-x)^{\frac{1}{3}}=q$

$\therefore$ The above equation is $p^2-3pq+2q^2=0$ or, $(p-2q)(p-q)=0$

or, $p=2q$ or, $(a+x)^{\frac{1}{3}}=2(a-x)^{\frac{1}{3}}$

or, $a+x=8(a-x)$ or, $x=\dfrac{7a}{9}$

when $p=q$, $x=0$

$\therefore$ The required solution of the above equation is $x=0,\ \dfrac{7a}{9}$

Indices

If a certain non-zero real or imaginary number a is multiplied m (positive integers) times in succession then the continued product so obtained is called the mth power of a and is written as a^m (read as, a to the power m). a is called the base $a\neq0$ and m is called the index or exponent of a^m

For example, $x^6=x\cdot x\cdot x\cdot x\cdot x\cdot x$

Laws of Indices

If a and b are two non-zero real numbers and m, n are positive integers then

(i) $a^m \cdot a^n = a^{m+n}$

(ii) (a) $a^m \div a^n = a^{m-n}$ (where $m > n$) (b) $a^m \div a^n = \frac{1}{a^{n-m}}$, (where $m < n$)

(iii) $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$ (iv) $a^0 = 1$, $a^{-1} = \frac{1}{a}$, $a^{-m} = \frac{1}{a^m}$, $a \neq 0$

(v) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(vi) $(ab)^m = a^m \cdot b^m$ (vii) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $[b \neq 0]$

(viii) If $a^m = b^m$ then $a = b$ when $m \neq 0$ (ix) If $a^m = a^n$ then $m = n\left(a \neq 0, 1, \pm\infty\right)$

(x) If a and b are two real numbers such that $a^n = b$, then a is called nth root of b.

(a) If $b > 0$, then there exists a unique positive nth root of b.

(b) If $b < 0$ and n is odd, then there is no positive nth root of a, but a has a unique negative nth root.

(c) If $a < 0$ and n is even, then there does not exist any real number b such that $b^n = a$.

(xi) $m < n \Rightarrow \begin{cases} a^m < a^n, & \text{if } a > 1 \\ a^m > a^n, & \text{if } 0 < a < 1 \end{cases}$

Note : 0^0 = undefined.

☞ $1^4 = 1^5$ but $4 \neq 5$.

$0^5 = 0^7$ but $5 \neq 7$

$(-1)^3 = (-1)^7$ but $3 \neq 7$

☞ The result $a^m b^m = (ab)^m$, is not always true.

Concept Illustrator

1. Simplify $4^{\frac{1}{3}} \times \left[2^{\frac{1}{3}} \times 3^{\frac{1}{2}}\right]^7 \div 9^{\frac{1}{4}}$

Sol. Given expression $= \left(2^2\right)^{\frac{1}{3}} \times \left(2^{\frac{1}{3}}\right)^7 \times \left(3^{\frac{1}{2}}\right)^7 \div \left(3^2\right)^{\frac{1}{4}}$

$= 2^{\frac{2}{3}} \times 2^{\frac{7}{3}} \times 3^{\frac{7}{2}} \div 3^{\frac{1}{2}} = 2^{\frac{2}{3}+\frac{7}{3}} \times 3^{\frac{7}{2}-\frac{1}{2}}$

$= 2^{\frac{9}{3}} \times 3^{\frac{6}{2}} = 2^3 \times 3^3 = 8 \times 27 = 216$

2. Simplify $\left\{\frac{25^{m+\frac{1}{4}}\cdot\sqrt{5.5^m}}{5\cdot\sqrt{5^{-m}}}\right\}^{\frac{1}{m}}$

Sol. Given expression $=\left\{\frac{(5^2)^{m+\frac{1}{4}}\cdot(5^1\cdot5^m)^{\frac{1}{2}}}{5\cdot(5^{-m})^{\frac{1}{2}}}\right\}^{\frac{1}{m}}$

$=\left\{\frac{5^{2m+\frac{1}{2}}\cdot(5^{1+m})^{\frac{1}{2}}}{5^1\cdot5^{-\frac{m}{2}}}\right\}^{\frac{1}{m}}=\left\{\frac{5^{2m+\frac{1}{2}}\cdot5^{\frac{1+m}{2}}}{5^{1-\frac{m}{2}}}\right\}^{\frac{1}{m}}=\left\{5^{2m+\frac{1}{2}+\frac{1+m}{2}-1+\frac{m}{2}}\right\}^{\frac{1}{m}}=\left(5^{3m}\right)^{\frac{1}{m}}$

$=5^{3m\times\frac{1}{m}}=5^3=125$

3. Simplify $\left(\frac{x^b}{x^c}\right)^{\frac{1}{bc}}\times\left(\frac{x^c}{x^a}\right)^{\frac{1}{ca}}\times\left(\frac{x^a}{x^b}\right)^{\frac{1}{ab}}$

Sol. Given expression $=\left(x^{b-c}\right)^{\frac{1}{bc}}\times\left(x^{c-a}\right)^{\frac{1}{ca}}\times\left(x^{a-b}\right)^{\frac{1}{ab}}$

$=x^{(b-c)\times\frac{1}{bc}}\times x^{(c-a)\times\frac{1}{ca}}\times x^{(a-b)\times\frac{1}{ab}}$

$=x^{\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}}=x^{\frac{b}{bc}-\frac{c}{bc}+\frac{c}{ca}-\frac{c}{ca}+\frac{a}{ab}-\frac{b}{ab}}=x^{\frac{1}{c}-\frac{1}{b}+\frac{1}{a}-\frac{1}{c}+\frac{1}{b}-\frac{1}{a}}=x^0=1$

4. If $x=0.6$, then find the value of $\left[1-\left\{1-\left(1-x^5\right)^{-1}\right\}^{-1}\right]^{\frac{-2}{5}}$

Sol. Given expression $\left[1-\left\{1-\left(1-x^5\right)^{-1}\right\}^{-1}\right]^{\frac{-2}{5}}=\left[1-\left\{1-\frac{1}{1-x^5}\right\}^{-1}\right]^{-\frac{2}{5}}$

$=\left[1-\left\{\frac{1-x^5-1}{1-x^5}\right\}^{-1}\right]^{\frac{-2}{5}}=\left[1-\left\{\frac{x^5}{x^5-1}\right\}^{-1}\right]^{\frac{-2}{5}}$

$=\left[1-\frac{x^5-1}{x^5}\right]^{\frac{-2}{5}}=\left[\frac{x^5-x^5+1}{x^5}\right]^{\frac{-2}{5}}=\left[\frac{1}{x^5}\right]^{\frac{-2}{5}}=\left[x^{-5}\right]^{-\frac{2}{5}}=x^{-5\times\left(\frac{-2}{5}\right)}=x^2=(0.6)^2=0.36$

5. If $a^x = m,\ a^y = n$ and $a^2 = \left(m^y n^x\right)^z$ show that, $xyz = 1$

Sol. $\because\ a^x = m \quad \therefore\ \left(a^x\right)^y = m^y \quad$ or, $a^{xy} = m^y$

Again, $a^y = n \quad \therefore\ \left(a^y\right)^x = n^x \quad$ or, $a^{xy} = n^x$

Now $\left(m^y n^x\right)^z = a^2 \quad$ or, $\left(a^{xy} \cdot a^{xy}\right)^z = a^2$

or, $\left(a^{xy+xy}\right)^z = a^2 \quad$ or, $\left(a^{2xy}\right)^z = a^2 \quad$ or, $a^{2xyz} = a^2$

$\therefore\ 2xyz = 2 \quad$ or, $xyz = 1$ (Proved)

6. If $\left(a^{n^2}\right)^n = \left(a^{2^n}\right)^2$ show that, $\sqrt[n+1]{n^3} = 2$

Sol. $\left(a^{n^2}\right)^n = \left(a^{2^n}\right)^2$

or, $a^{n^2 \cdot n} = a^{2^n \cdot 2} \quad \therefore\ n^2 \cdot n = 2^n \cdot 2^1 \quad$ or, $n^3 = 2^{n+1}$

$\therefore\ \sqrt[n+1]{n^3} = 2$ (Proved)

7. If $\left(\frac{y}{z}\right)^a \left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = 1$. Show $\left(\frac{y}{z}\right)^{\frac{1}{b-c}} = \left(\frac{z}{x}\right)^{\frac{1}{c-a}} = \left(\frac{x}{y}\right)^{\frac{1}{a-b}}$

Sol. $\left(\frac{y}{z}\right)^a \left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = 1 \quad$ or, $\left(\frac{y}{z}\right)^a \left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = \left(\frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y}\right)^a \quad \left[\because (1)^a = 1\right]$

or, $\left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = \left(\frac{z}{x}\right)^a \cdot \left(\frac{x}{y}\right)^a \quad$ or, $\left(\frac{x}{y}\right)^{c-a} = \left(\frac{z}{x}\right)^{a-b}$

or $\left(\frac{x}{y}\right)^{\frac{1}{a-b}} = \left(\frac{z}{x}\right)^{\frac{1}{c-a}} \quad \cdots(1)$

Similarly, $\left(\frac{y}{z}\right)^a \cdot \left(\frac{z}{x}\right)^b \cdot \left(\frac{x}{y}\right)^c = \left(\frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y}\right)^b$

i.e. $\left(\frac{x}{y}\right)^{b-c} = \left(\frac{y}{z}\right)^{a-b} \quad$ or, $\left(\frac{x}{y}\right)^{\frac{1}{a-b}} = \left(\frac{y}{z}\right)^{\frac{1}{b-c}} \quad \cdots(2)$

From (1) and (2) $\left(\frac{y}{z}\right)^{\frac{1}{b-c}} = \left(\frac{z}{x}\right)^{\frac{1}{c-a}} = \left(\frac{x}{y}\right)^{\frac{1}{a-b}}$

8. If $(111.1)^a = (11.11)^b = (1.111)^c$, show $\frac{b}{a} + \frac{b}{c} = 2$

Sol. Let $(111.1)^a = (11.11)^b = (1.111)^c = k$

$\therefore$ $111.1 = k^{\frac{1}{a}}$, $11.11 = k^{\frac{1}{b}}$, $1.111 = k^{\frac{1}{c}}$

Now, $k^{\frac{1}{a}} \cdot k^{\frac{1}{c}} = 111.1 \times 1.111 = (11.11)^2 = k^{\frac{2}{b}}$

or, $k^{\frac{1}{a}+\frac{1}{c}} = k^{\frac{2}{b}}$

$\therefore \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

9. If $x = 3 + 3^{\frac{2}{3}} + 3^{\frac{1}{3}}$, find the value of $x^3 - 9x^2 + 18x - 12$

Sol. $\because$ $x = 3 + 3^{\frac{2}{3}} + 3^{\frac{1}{3}}$ $\therefore$ $x - 3 = 3^{\frac{2}{3}} + 3^{\frac{1}{3}}$

or, $(x-3)^3 = \left(3^{\frac{2}{3}} + 3^{\frac{1}{3}}\right)^3$ [cubing both sides]

or, $x^3 - 3^3 - 3 \cdot x \cdot 3(x-3) = \left(3^{\frac{2}{3}}\right)^3 + \left(3^{\frac{1}{3}}\right)^3 + 3 \cdot 3^{\frac{2}{3}} \cdot 3^{\frac{1}{3}}\left(3^{\frac{2}{3}} + 3^{\frac{1}{3}}\right)$

or, $x^3 - 27 - 9x^2 + 27x = 3^2 + 3 + 3^{1+\frac{2}{3}+\frac{1}{3}}(x-3)$ $\left[\because 3^{\frac{2}{3}} + 3^{\frac{1}{3}} = x - 3\right]$

or, $x^3 - 9x^2 + 27x - 27 = 12 + 9(x-3) = 9x - 15$ or, $x^3 - 9x^2 + 27x - 9x - 27 + 15 = 0$

or, $x^3 - 9x^2 + 18x - 12 = 0$

$\therefore$ the value of the given expression $= 0$

10. If $(7.77)^x = (0.777)^y = 1000$, then show that, $\frac{1}{x} - \frac{1}{y} = \frac{1}{3}$

Sol. Given that : $(7.77)^x = (0.777)^y = 1000 = 10^3$

$\therefore$ $(7.77)^x = 10^3 \Rightarrow 7.77 = 10^{\frac{3}{x}}$

Again, $(0.777)^y = 10^3 \Rightarrow 0.777 = 10^{\frac{3}{y}}$

Now from (1) & (2), we have $\frac{7.77}{0.777} = \frac{10^{\frac{3}{x}}}{10^{\frac{3}{y}}} = 10^{\frac{3}{x}-\frac{3}{y}} \Rightarrow 10^{\frac{3}{x}-\frac{3}{y}} = 10^1$

$\Rightarrow \frac{3}{x} - \frac{3}{y} = 1 \Rightarrow \frac{1}{x} - \frac{1}{y} = \frac{1}{3}$ (Proved)

11. If $2^n = 4^y = 8^z$ and $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=\frac{22}{7}$

Sol. If $2^x = 4^y = 8^z \Rightarrow 2^x = 2^{2y} = 2^{3z} \Rightarrow x = 2y = 3z$ then find the values of x, y, z.

$\therefore \frac{x}{6}=\frac{y}{3}=\frac{z}{2}=k$ (say), $\quad \therefore \; x = 6k,\; y = 3k,\; z = 2k$

Again, $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=\frac{22}{7} \Rightarrow \frac{1}{2\cdot 6k}+\frac{1}{4\cdot 3k}+\frac{1}{8\cdot 2k}=\frac{22}{7}$

$\Rightarrow \frac{4+4+3}{48k}=\frac{22}{7} \Rightarrow \frac{11}{48k}=\frac{22}{7} \Rightarrow k=\frac{77}{22\times 48}=\frac{7}{96}$

$\therefore \; x = 6k = \frac{6\times 7}{96}=\frac{7}{16},\; y = 3k = \frac{3\times 7}{96}=\frac{7}{32},\; z = 2k = \frac{2\times 7}{96}=\frac{7}{48}$

13. Prove that $(x+y)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)=\frac{x^{2^n}-y^{2^n}}{x-y}$

$$\text{L.H.S} = \frac{(x-y)(x+y)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

$$= \frac{(x^2-y^2)(x^2+y^2)(x^4+y^4)\cdots\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}$$

Thus, multiplying upto the last term, we get

$$\text{L.H.S.} = \frac{\left(x^{2^{n-1}}-y^{2^{n-1}}\right)\left(x^{2^{n-1}}+y^{2^{n-1}}\right)}{x-y}=\frac{\left(x^{2^{n-1}}\right)^2-\left(y^{2^{n-1}}\right)^2}{x-y}=\frac{x^{2^n}-y^{2^n}}{x-y}$$

14. If $pqr = 1$ show that,

$$\frac{1}{1+p+q^{-1}}+\frac{1}{1+q+r^{-1}}+\frac{1}{1+r+p^{-1}}=1$$

Sol. We have, $\frac{1}{1+p+q^{-1}}=\frac{p^{-1}}{p^{-1}\left(1+p+q^{-1}\right)}=\frac{p^{-1}}{p^{-1}+p^0+\frac{1}{pq}}$

$= \frac{p^{-1}}{p^{-1}+1+r} \qquad \left[\because pqr = 1 \;\therefore \frac{1}{pq}=r\right]$

$= \frac{p^{-1}}{1+r+p^{-1}}$

Again, $\frac{1}{1+q+r^{-1}}=\frac{r}{r\left(1+q+r^{-1}\right)}=\frac{r}{r+qr+r^0}$

$= \frac{r}{r+p^{-1}+1} \qquad \left[\because \; pqr = 1 \;\therefore\; qr=\frac{1}{p}=p^{-1}\right]$

$= \frac{r}{1+r+p^{-1}}$

$\therefore$ L.H.S. $= \frac{p^{-1}}{1+r+p^{-1}}+\frac{r}{1+r+p^{-1}}+\frac{1}{1+r+p^{-1}}=\frac{p^{-1}+r+1}{1+r+p^{-1}}=1$ (Proved)

15. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ then prove that $\frac{1}{a^{2m+1}}+\frac{1}{b^{2m+1}}+\frac{1}{c^{2m+1}}=\frac{1}{a^{2m+1}+b^{2m+1}+c^{2m+1}}$

Sol. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \Rightarrow (a+b)(b+c)(c+a)=0$ (prove) $\therefore$ at least one factor is zero,

That is a $a=-b$ or $b=-c$ or $c=-a$ $\quad\therefore\ a^{2m+1}=-b^{2m+1}$ or $b^{2m+1}=-c^{2m+1}$

$$\frac{1}{a^{2m+1}}+\frac{1}{b^{2m+1}}+\frac{1}{c^{2m+1}}$$

$$=\frac{1}{a^{2m+1}}-\frac{1}{a^{2m+1}}+\frac{1}{c^{2m+1}}=\frac{1}{0+c^{2m+1}}=\frac{1}{a^{2m+1}+b^{2m+1}+c^{2m+1}}$$

Try Your Self

NCERT BOARD

1. Find six rational numbers between 3 and 4.
2. Find five rational numbers between $\frac{3}{5}$ and $\frac{4}{5}$
3. State whether the following statements are true or false. Give reasons for your answers.
   (i) Every natural number is a whole number.
   (ii) Every integer is a whole number.
   (iii) Every rational number is a whole number.
4. Locate $\sqrt{3}$ on the number line
5. State whether the following statements are true or false. Justify your answers.
   (i) Every irrational number is a real number.
   (ii)Every point on the number line is of the form $\sqrt{m}$. where m is a natural number.
   (iii) Every real number is an irrational number.
6. Are the square roots of all positive integers irrational? If not, give an example of the square root of a number that is a rational number.
7. Show how $\sqrt{5}$ can be represented on the number line
8. Write the following in decimal form and say what kind of decimal expansion each has :
   (i) $\frac{36}{100}$ (ii) $\frac{1}{11}$ (iii) $4\frac{1}{8}$
   (iv) $\frac{3}{13}$ (v) $\frac{2}{11}$ (vi) $\frac{329}{400}$
9. You know that $\frac{1}{7}=0.\overline{142857}$. Can you predict what the decimal expansions of $\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}$ are without actually doing the long division? If so, how?

10. Express the following in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$

(i) $0.\overline{6}$ (ii) $0.4\overline{7}$ (iii) $0.\overline{001}$

11. Express 0.99999 ... in the form $\frac{p}{q}$. Are you surprised by your answer? With your teacher and classmates discuss why the answer makes sense.

12. What can the maximum number of digits be in the repeating block of digits in the decimal expansion of $\frac{1}{17}$? Perform the division to check your answer.

13. Look at several examples of rational numbers in the form $\frac{p}{q}(q \neq 0)$, where p and q are integers with no common factors other than l and having terminating decimal representations (expansions). Can you guess what property q must satisfy?

14. Write three numbers whose decimal expansions are non-terminating non-recurring.

15. Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.

16. Classify the following numbers as rational or irrational:

(i) $\sqrt{23}$ (ii) $\sqrt{225}$ (iii) 0.3796

(iv) $7.478478...$ (v) $1.101001000100001.....$

17. Classify the following numbers as rational or irrational :

(i) $2-\sqrt{5}$ (ii) $\left(3+\sqrt{23}\right)-\sqrt{23}$ (iii) $\frac{2\sqrt{7}}{7\sqrt{7}}$

(iv) $\frac{1}{\sqrt{2}}$ (v) 2π

18. Simplify each of the following expressions :

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

19. Recall, π is defined as the ratio of the circumference (say c) of a circle to its diameter (say d) That is $\pi = \frac{c}{d}$. This seems to contradict the fact that π is irrational. How will you resolve this contradiction?

20. Rationalise the denominators of the following :

(i) $\frac{1}{\sqrt{7}}$ (ii) $\frac{1}{\sqrt{7}-\sqrt{6}}$ (iii) $\frac{1}{\sqrt{5}+\sqrt{2}}$ (iv) $\frac{1}{\sqrt{7}-2}$

NUMBER SYSTEM

Fill in the Blanks

1. Rational numbers and irrational numbers together constitute a set of __________.
2. Simplified form of $(2\sqrt{5}+3\sqrt{2})^2$ is ____________.
3. After rationalising the denominator of $\frac{\sqrt{3}+1}{2\sqrt{2}-\sqrt{3}}$ we get __________.
4. The product of a non-zero rational number and an irrational number will be ____________.
5. $2^{-4}(2\sqrt{3})^2 =$ _________.
6. If $x=\frac{\sqrt{2}+1}{\sqrt{2}-1}$, then $\frac{1}{x^2}$ = _________.
7. When $\sqrt{x^{-2}y^3}$ is written in exponential form, it is equal to _________.
8. Every real number has a unique point on the______________.
9. The value of $\frac{2}{\sqrt{3}}$ approximately upto 3 decimal places where $\sqrt{3}=1.732$ ____________.
10. $\left(\frac{\sqrt{3}}{8}\right)\div\left(\frac{\sqrt{3}}{8}\right)^5$ =______________.

Match The Followings

Part A	Part B
The numbers $7\sqrt{3}, \frac{7}{\sqrt{5}}, \pi-2$ are	$\frac{\sqrt{5}+\sqrt{3}}{2}$
$\left(7+3\sqrt{2}\right)\left(7-3\sqrt{2}\right)=$	terminating
On number line, negative real numbers lie on the	$a^2 - b$
This is the rationalised form of $\frac{1}{\sqrt{5}-\sqrt{3}}$	irrational numbers
$125^{-1/3}$ is same as this rational number.	3
Decimal expansion of $4\frac{1}{8}$ is	$\frac{1}{28^3}$
Product of $2^{1/5}$ and $16^{1/5}=$	31

This is the value of x which satisfies the equation $\left(\frac{a}{b}\right)^{x-1} = \left(\frac{b}{a}\right)^{x-5}$	2
$28^2 . 28^{-5}$ is same as	left side of zero
$\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right)$ is equal to	$\frac{1}{5}$

True False

1. $\left(\sqrt{2}+2\right)^2$ is a rational number.

2. Addition, subtraction, multiplication and division of two irrational numbers may or may be irrational.

3. The number 0.318456318456318456.... is an irrational number.

4. The values of a and b in $\frac{3+\sqrt{7}}{3-\sqrt{7}} = a+\sqrt{7}$ are 8 and 3 respectively.

5. π and e are irrational numbers.

6. To rationalise $\frac{1}{3-\sqrt{7}}$, we multiply this by $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

7. The decimal expansion of an irrational number is neither a terminating nor repeating decimal.

8. 4.6734467344... is a rational number.

9. Irrational numbers cannot be represented on a number line.

10. $\frac{a^m}{b^m} = (ab)^m$

Short Answers

1. Represent 14.3222…. as a rational number.

2. Find the values of a and b in $\frac{5+2\sqrt{3}}{7+4\sqrt{3}} = a+b\sqrt{3}.$

3. Express 0.545454… in the form of $\frac{p}{q}$.

4. Is $(\sqrt{5}+\sqrt{7}$ an irrational number? If yes, then prove.

MCQS

1. Which of the following value of x is an irrational number?

A. $x^2 = 0.81$

B. $x^2 = \frac{15}{6}$

C. $x^2 = 0.0064$

D. $x^2 = 9$

2. What is the cube root of rational number -5^{15} ?
A. −625
B. −25
C. −125
D. −3,125

3. To rationalise the denominator of the fraction $\frac{\sqrt{3}+4\sqrt{2}}{4\sqrt{3}-\sqrt{2}}$, we will multiply and divide it by which of the following rationalising factors?

A. $4\sqrt{3}-\sqrt{2}$

B. $-4\sqrt{3}+\sqrt{2}$

C. $\sqrt{3}-\sqrt{2}$

D. $4\sqrt{3}+\sqrt{2}$

4. Which of the following are an approximate equivalent rational value of an irrational number π?

A. $\frac{11}{14}$

B. $\frac{20}{14}$

C. $\frac{44}{14}$

D. $\frac{35}{14}$