Complex Numbers

MATHEMATICS

For IIT / JEE

Definitions of Complex Numbers and its position in number :

Deffinitions : An expression of the form $(a + ib)$, where a and b are both real and $i = \sqrt{-1}$ is called a complex number. Where i is called iota. Here the sign '+' does not indicate addition as usually understood. It is a mare symbol.

The real part of the complex number $(a + ib)$ is a and is written as $Re(a + ib)$, while its imaginary part is b and is written as Im $(a + ib)$. A complex number is usually denoted by z.

If $a = 0$, then the complex number $(a + ib)$ becomes ib, which is purely imaginary. If $b = 0$, then the complex number $(a + ib)$ becomes purely real. If both $a = 0$ and $b = 0$, then the complex number becomes zero. Hence the real numbers are particular cases of complex numbers.

If the real parts of two complex numbers be the same and their imaginary parts be same but of opposite signs, then the two numbers are said to be complex conjugate numbers. Thus $(a + ib)$ and $(a - ib)$ are complex conjugate numbers. If z be a complex number $(a + ib)$, then its conjugate $(a - ib)$ is denoted by $\bar{z}$. It is evident that conjugate of $\bar{z}$ is z.

A complex number $z = a + ib$ may also be written as (a, b) (ordered pair of real numbers a and b).

Note

(i) $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

(ii) $(a, b) + (c, d) = (a + c, b + d)$ (iii) $(a, b), (c, d) = (ac - bd, ad + bc)$,

where (c, d) is another complex number.

The first component a of the complex number (a, b) is its real part, while the second component b is its imaginary part.

Thus the complex number $(a, 0)$ is purely real, $(0, b)$ is purely imaginary and $(0, 1)$ is the fundamental imaginary unit.

It is observed that the two conventions $a + ib$ and (a, b) are equvalent, if we simply replace i^2 by (-1) where necessary.

Properties of complex numbers :

(a) If $a + ib = 0$, a and b being real, then $a = 0$ and $b = 0$.(b) If $a + ib = c + id$, then $a = c$ and $b = d$.

Concept Illustrator

Qu. If x, y are real and $x + iy = -i(-2 + 3i)$, find x and y.

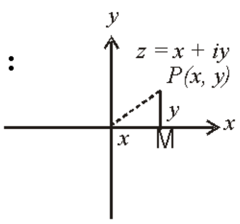
Sol. $x + iy = 2i - 3i^2 = 2i + 3$ or, $x = 3,\ y = 2$

Note Inequality relation does not hold good in case of complex numbers having non-zero imaginary parts. For example, the statement $3 + 2i > 7 + 5i$ makes no sense.

(c) The algebraic sum, difference, product or ratio of two complex numbers is a complex number.

(d) The sum and the product of two conjugate complex numbers are both real.

Argand Plane and Geometrical Representation of Complex Numbers :



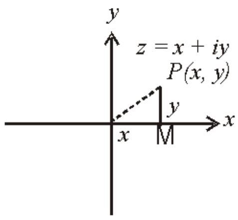
Let O be the origin and Ox, Oy be the x-axis and y-axis respectively

Then, any complex number $z = x + iy = (x, y)$ may be represented by

a unique point P whose coordinates are (x, y). The representation of

complex numbers as point in a plane forms as Argand diagram.
The plane on which complex numbers are represented is known as the complex plane or Argand's plane or gaussion plane.
x-axis is called then real axis and y-axis the imaginary axis.
The complex number $z = x + iy$ is known as the affix of the point (x, y) which it represents.

Modulus of Complex Number :



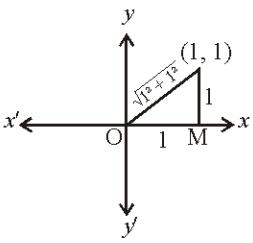
Let $z = x + iy,\ x, y \in R(y \neq 0)$ be a complex number.
The length of the line segment OP is called the modulus of z and is denoted by $|z|$. Fig. (a), we have $OP^2 = OM^2 + MP^2$

$\Rightarrow OP^2 = x^2 + y^2 \Rightarrow OP = \sqrt{x^2 + y^2}$.

Thus, $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$

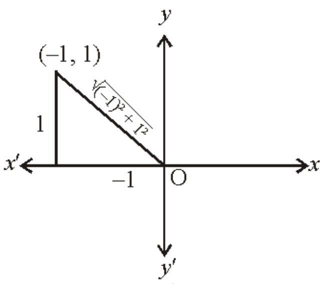
Concept Illustrator

❖ If $z = 1 + i$ then $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$



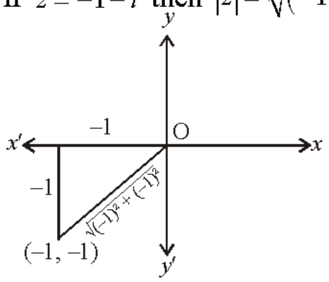
$z = 1 + i$

❖ If $z = -1 + i$ then $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$



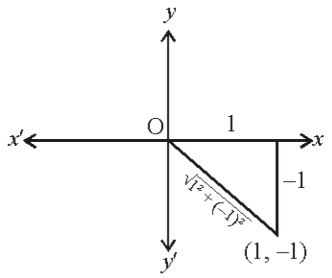
$z = -1 + i$

❖ If $z = -1 - i$ then $|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$



$z = -1 - i$

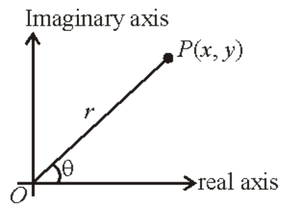
❖ If $z = 1 - i$ then $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$



$z = 1 - i$

Certesian representation of complex numbers, their modulus and amplitude :

(a) Cartesian Form (Geometric Representation) : To each complex number there corresponds one and only one point in plane, and conversely to each point in theh plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.



Every complex number $z = x + iy$ can be represented by a point on Cartesian jplane known as complex plane (Argand diagram) by the ordered pair (x, y).

Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called the argument of amplitude.

$|z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \left(\frac{y}{x}\right)$ (angle made by OP with positive x-axis)

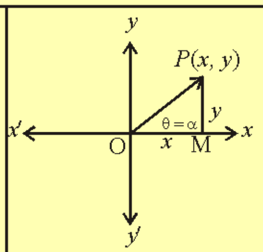
Note

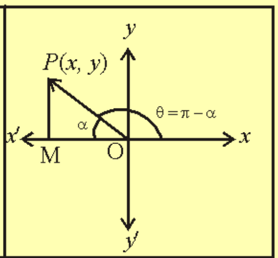
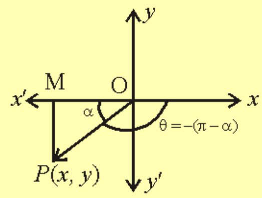
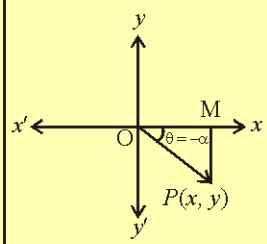
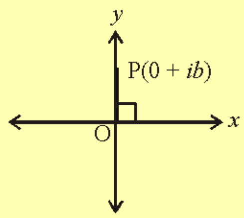
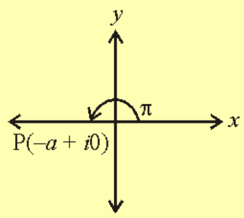
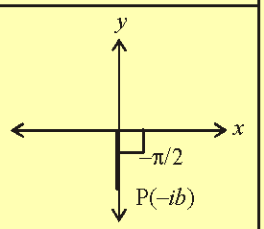
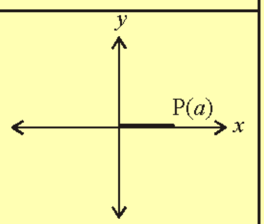
(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta;\ n \in I$ will also be the argument of that complex number. Any two consecutive arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(iii) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is only given by its modulus.

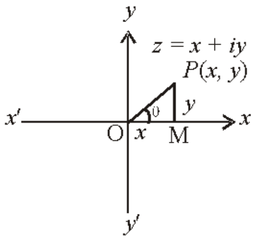
The argument of z depends upon the quadrant in which the point P lies as discussed below :

(i) $z = x + iy$, z lies in first quadrant ($x > 0$ and $y > 0$) From the figure $\tan\alpha = |y/x|$ and $\theta = \alpha$. Then $\arg(z) = \tan^{-1}|y/x|$.	

(ii) $z = x + iy$, z lies in second quadrant ($x < 0$ and $y > 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = \pi - \alpha$. Then $\arg(z) = \pi - \alpha$, where α is the acute angle given by $\tan^{-1}\|y/x\|$.	
(iii) $z = x + iy$, z lies in third quadrant ($x < 0$ and $y < 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = (\pi - \alpha) = -\pi + \alpha$. Then, $\arg(z) = \alpha - \pi$ where α is the acute angle given by $\tan\alpha = \|y/x\|$.	
(iv) $z = x + iy$, z lies in fourth quadrant ($x > 0$ and $y < 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = -\alpha$. Then $\arg(z) = -\alpha$, where α is the acute angle given by $\tan\alpha = \|y/x\|$.	
(v) $z = x + iy$, z lies in x-axis ($x = 0$ and $y > 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = \pi/2$. Then principal value $\arg(z) = \pi/2$.	
(vi) $z = x + iy$, z lies in negative direction of a x-axis ($x < 0$ and $y = 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = \pi$. Then principal value $\arg(z) = \pi$.	
(vii) $z = x + iy$, z lies in negative direction of a y-axis ($x = 0$ and $y < 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = \pi$. Then principal value $\arg(z) = -\pi/2$.	
(viii) $z = x + iy$, z lies in positive direction of a x-axis ($x > 0$ and $y = 0$) From the figure $\tan\alpha = \|y/x\|$ and $\theta = 0$. Then principal value $\arg(z) = 0$.	

Polar representation of complex numbers, their modulus and amplitude (restrictions on 'R' and 'θ') :

Let O be the origin and Ox and Oy be the x-axis and y-axis respectively. Let $z = x + iy$ be a complex number represented by the point $P(x, y)$.



Draw $PM \perp Ox$. Then $OM = x$ and $PM = y$. Join OP.

Let OP be r and $\angle XOP = \theta$

Then $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore z = x + iy = r(\cos\theta + i\sin\theta)$

Comparing real and imaginary parts, we get $x = r\cos\theta$ ......(i) and $y = r\sin\theta$ ......(ii)

Squaring (i) and (ii) and adding, we get $r^2 = x^2 + y^2$ or, $r = \sqrt{x^2 + y^2} = |z|$

Also, $\tan\theta = \frac{y}{x}$

Case I : Polar form of $z = x + iy$ where $x > 0$ and $y > 0$.

In this case, we have $\theta = \alpha$.

So, the polar form of $z = x + iy$ is $z = r(\cos\alpha + i\sin\alpha)$

Case II : Polar form of $z = x + iy$ when $x < 0$ and $y > 0$.

In this case, we have $\theta = \pi - \alpha$.

So, the polar form of $z = x + iy$ is $z = r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)] = r(-\cos\alpha + i\sin\alpha)$.

Case III : Polar form of $z = x + iy$ when $x < 0$ and $y < 0$.

In this case, we have $\theta = -(\pi - \alpha)$.

So, the polar form of z is given by $z = r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))] \Rightarrow z = r(-\cos\alpha - i\sin\alpha)$.

Case IV : Polar form of $z = x + iy$ when $x > 0$ and $y < 0$.

In this case, we have $\theta = -\alpha$.

So, the polar form of z is $z = r[\cos(-\alpha) + i\sin(-\alpha)] = r[\cos\alpha - i\sin\alpha]$.

Euler's Form of Complex Number :

$e^{i\theta} = \cos\theta + i\sin\theta$

Any complex number can be expresed as $z = x + iy$ (Cartesian form)

$= |z|[\cos\theta + i\sin\theta]$ (polar form) $= |z|e^{i\theta}$

Product of Two Complex Numbers :

Let two complex numbers be $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$. Now, $z_1 z_2 = |z_1|e^{i\theta_1} \times |z_2|e^{i\theta_2}$

$= |z_1||z_2|e^{i(\theta_1+\theta_2)} = |z_1||z_2|[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Thus, $|z_1 z_2| = |z_1||z_2|$ $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Division of Two Complex Numbers :

$$\frac{z_1}{z_2}=\frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}}=\frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)} \Rightarrow \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right)=\theta_1-\theta_2=\arg(z_1)-\arg(z_2)$$

Logarithm of a complex number is given by $\log_e(x+iy)=\log_e\left(|z|e^{i\theta}\right)$

$$=\log_e|z|+\log_e e^{i\theta}=\log_e|z|+i\theta=\log_e\sqrt{(x^2+y^2)}+i\arg(z)$$

$$\therefore\ \log_e(z)=\log_e|z|+i\arg(z)$$

Concept Illustrator

Qu. Express $\left(\sqrt{3}-i\right)$ in modulus-amplitude form.

Sol. Let $z=\sqrt{3}-i$

If $|z|=r$ and amp $z=\theta(-\pi<\theta\leq\pi)$ then, the modulus-amplitude form of z is $z=r(\cos\theta+\sin\theta)$

Now, $|z|=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=\sqrt{4}=2$

Again, in the z-plane, the point $z=\sqrt{3}-i=\left(\sqrt{3},-1\right)$ lies in the 4th quadrant ; hence, if amp $z=\theta$

then, $\tan\theta=\frac{-1}{\sqrt{3}}$ where $\frac{-\pi}{2}<\theta<0\quad \therefore\ \tan\theta=\frac{-1}{\sqrt{3}}=\tan\left(\frac{-\pi}{6}\right)$

$\therefore\ \theta=\frac{-\pi}{6}$ or, amp $z=\frac{-\pi}{6}$

$\therefore$ modulus-amplitude form of $z=\left(\sqrt{3}-i\right)$ is,

$z=2\left[\cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right]$ [putting the values of r and θ in (1)] $=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$

Properties of Argument

(i) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$

(ii) $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

(iii) $\arg\left(\frac{z}{z}\right)=2\arg z$

(iv) $\arg\left(z^n\right)=n\arg z$

(v) If $\arg\left(\frac{z_2}{z_1}\right)=\theta$, then $\arg\left(\frac{z_1}{z_2}\right)=2k\pi-\theta$ where $k\in I$

(vi) $\arg\overline{z}=-\arg z$

Concept Illustrator

Qu. Find the argument of $z=-3-3i$

Sol. Arg. of $z=\tan^{-1}\left(\frac{-3}{-3}\right)=-\pi+\tan^{-1}1=-\pi+\frac{\pi}{4}=\frac{-3\pi}{4}$ $\left(\because z \text{ lies in } 3^{rd} \text{quadrant}\right)$

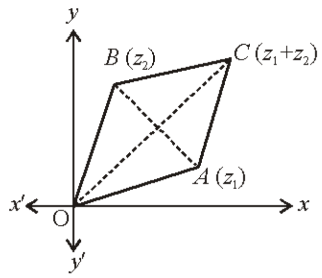
Triangle inequality, Properties of Modulus, Geometric significance of Modulus, Properties of amplitude :

In triangle OAC

$OC \le OA + AC$

$OA \le AC + OC$

$AC \le OA + OC$



Using these inequalities we have $\left||z_1|-|z_2|\right| \le |z_1+z_2| \le |z_1|+|z_2|$

Similarly from triangle OAB, we have $\left||z_1|-|z_2|\right| \le |z_1-z_2| \le |z_1|+|z_2|$

Note

(i) $\left||z_1|-|z_2|\right|=|z_1+z_2|$, $|z_1-z_2|=|z_1|+|z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(ii) $|z_1+z_2|=|z_1|+|z_2|$, $\left||z_1|-|z_2|\right|=|z_1-z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of the origin.

Concept Illustrator

Qu. If z_1 and z_2 be the two complex number so that $|z_1+z_2| \le |z_1|+|z_2|$

Sol. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$

$\therefore |z_1|=r_1$ and $|z_2|=r_2$

then, $z_1+z_2=r_1(\cos\theta_1+i\sin\theta_1)+r_2(\cos\theta_2+i\sin\theta_2)$

$=(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)$

$\therefore |z_1+z_2|^2=(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2$

$=r_1^2+r_2^2+2r_1r_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)$

$=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2) \le r_1^2+r_2^2+2r_1r_2$ $\left[\because \cos(\theta_1-\theta_2) \le 1\right]$

$=(r_1+r_2)^2$ or, $|z_1+z_2| \le r_1+r_2$ $\left[\because |z_1+z_2|, \text{positive formula } r_1 \text{ and } r_2\right]$

$\therefore |z_1+z_2| \le |z_1|+|z_2|$

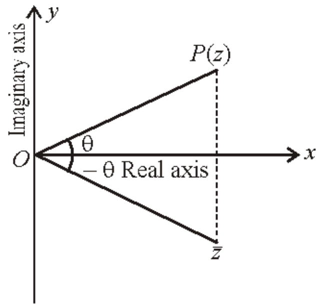
Conjugate of complex number, Properties of conjugate and its geometric significance :

For complex number $z = x + iy, (x, y) \in R,$ its conjugate is defined as $\overline{z} = x - iy$. Clearly, $z = x + iy$ is represented by a point $P(x, y)$ in the Argand plane. Now, $z = x + iy \Rightarrow \overline{z} = x - iy = x + i(-y)$

So, $\overline{z}$ is represented by a point $Q(x, -y)$ is the Argand plane. Clearly, Q is the image of point P on the real axis.

Thus, if a point P represents a complex number z, then its conjugate $\overline{z}$ is represented by the image of P on the real axis.

It is evident from the following figure that $|z| = |\overline{z}|$ and $\arg(\overline{z}) = -\arg(z)$.



Also, we have $\text{Re}(z) = \dfrac{(z + \overline{z})}{2}$ and $\text{Im}(z) = \dfrac{(z - \overline{z})}{2i}$. Thus if $z = |z|e^{i\theta}$, then $\overline{z} = |z|e^{-i\theta}$.

Conjugate of a complex number $z = a + ib$ is defined as $\overline{z} = a - ib$.

For example, $z = 4 + 5i$ and $\overline{z} = 4 - 5i$

Properties of Conjugate

1. $(\overline{\overline{z}}) = z$
2. $z = \overline{z}$ if and only if z is purely real
3. $z = -\overline{z}$ if and only if z is purely imaginary
4. $z + \overline{z} = 2\text{Re}(z)$ and $z = \overline{z} = 2i\,\text{Im}(z)$
5. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
6. $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$
7. $\overline{z_1 z_2} = \overline{z}_1 . \overline{z}_2$
8. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2},\ z_2 \neq 0$
9. If $z = f(z_1)$, then $\overline{z} = f(\overline{z_1})$
10. $(\overline{z^n}) = (\overline{z})^n$
11. $z_1\overline{z}_2 + \overline{z}_1 z_2 = 2\text{Re}(\overline{z}_1 z_2) = 2\text{Re}(z_1\overline{z}_2)$

The complex number $\dfrac{a + ib}{c + id}$ in the form $A + iB$

We have, $\dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$

[Multiplying the Nu. and the Dn. by the conjugate of the Dn.]

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$= A + iB, \text{ where } A = \frac{ac+bd}{c^2+d^2} \text{ and } B = \frac{bc-ad}{c^2+d^2}.$$

Important remark

To put the complex number $\frac{a+ib}{c+id}$ in the for $A+iB$ we should multiply the numerator and the denomiantor by the conjugate of the denominator.

Equation of different geometrical figure in terms of complex :

Equation of Staraight Line :

Equation of straight line through z_1 and z_2 is given by $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$ or, $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

The general equation of straight line is $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

Condition of collinearity :

(a) Three points z_1, z_2 and z_3 are collinear if,

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

(b) If three points A(z_1), B(z_2), C(z_3) are collinear then slope of AB = slope of BC = slope of AC

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

Equation of the perpendicular bisector :

The equation of the perpendicular bisector of the line segment joining points $A(z_1)$ and $A(z_2)$ is

$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

Length of perpendicular :

The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by

$$\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a| + |\bar{a}|} \quad or, \quad \frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$$

Equation of a circle :

The equation of a circle with centre z_0 and radius *r* is $|z - z_0| = r$.

The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where *b* is a real number and *a* is a complex number. The centre of this circle is '$-a$' and its radius is $\sqrt{a\bar{a} - b}$.

Note

(i) If the centre of the circle is at origin and radius r then its equation is $|z| = r$.

(ii) $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.

Key Results to Remember

1. $\left|\frac{z - z_1}{z - z_2}\right| = k$ is a circle if $k \neq 1$ and is a line if $k = 1$.

2. The equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

3. If $\arg\left[\frac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right] = \pm\pi,\ 0$, then the points z_1, z_2, z_3, z_4 are concyclic.

4. $|z - z_0| < r$ represents interior of the circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.

- **Equation of Ellipse :**

  If $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$, then the point z describes on ellipse having foci at z_1 and z_2 and $a \in R^+$.

- **Equation of Hyperbola :**

  If $|z - z_1| - |z - z_2| = 2a$, where $2a < |z_1 - z_2|$, then the point z describes a hyperbola having foci at z_1 and z_2 and $a \in R^+$.

- **Some Properties of Triangle :**

1. If z_1, z_2, z_3 are the vertices of triangle then centroid z_0 may be given as $z_0 = \frac{z_1 + z_2 + z_3}{3}$

2. If z_1, z_2, z_3 are the vertices of an equilateral triangle then the circumcentre z_0 may be given as $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

3. If z_1, z_2, z_3 are the vertices of an equilateral triangle then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

   or, $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

4. If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2 then $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

5. If z_1, z_2, z_3 are the vertices of right angled isosceles triangle then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Rotation theorem (Coni method), Problems on Coni Method :

i.e. angle between two intersecting lines.

Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.

Then we have $\overrightarrow{AC} = z_3 - z_1$ and $\overrightarrow{AB} = z_2 - z_1$

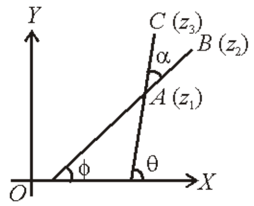
and let $\arg \overrightarrow{AC} = \arg(z_3 - z_1) = \theta$ & $\arg \overrightarrow{AC} = \arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$

$\because \angle CAB = \alpha = \theta - \phi = \arg \overrightarrow{AC} - \arg \overrightarrow{AB} = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

or, angle between AC and $Ab = \arg\left(\frac{\text{affix of } C - \text{affix of } A}{\text{affix of } B - \text{affix of } A}\right)$

For any complex number z we have $z = |z| e^{i(\arg z)}$



similarly $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i}\left(\arg \frac{z_3 - z_1}{z_3 - z_1}\right)$

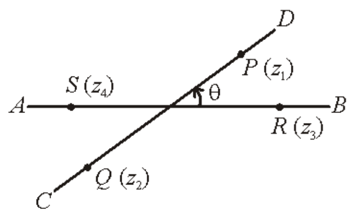
or, $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \frac{AC}{AB} e^{i\alpha}$

Note

1. Here only principal values of the arguments are considered.

2. $\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if AB coincides with CD, then

$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or, $\pm\pi$ so that, $\frac{z_1 - z_2}{z_3 - z_4}$ is real.



It follows that if $\frac{z_1 - z_2}{z_3 - z_4}$ is real, then the points A, B, C, D are collinear.

3. If AB is perpendicular to CD, then $\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$ so $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary.

Concept Illustrator

Qu. Show that the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argund plane is equilateral if and only $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$.

Sol. From Coni method $\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \frac{\pi}{3}$

$\therefore \frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC} e^{\frac{i\pi}{3}}$

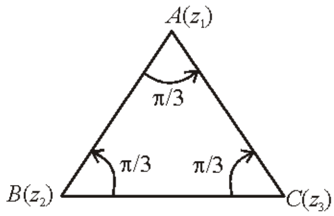
$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = e^{\frac{i\pi}{3}}$ ...... (1) $\quad (\because AB = BC)$

and $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{\pi}{3}$

$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \frac{CA}{AB} e^{\frac{i\pi}{3}} \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = e^{\frac{i\pi}{3}}$ .... (2) $\quad (\because CA = AB)$

from (1) and (2) we get

$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_1}{z_2 - z_1}$



$\Rightarrow (z_1 - z_2)^2 + (z_3 - z_1)(z_3 - z_2) = 0$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$

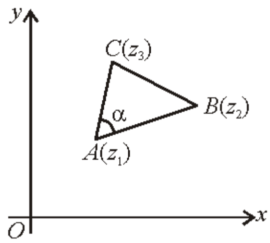
$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_3 - z_1)(z_1 - z_2) = 0$ .... (3)

Dividing equation (3) by $(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$ then we get $\frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} = 0$

It follows that if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k purely imaginary number, then AB and CD are perpendicular to each other.

If z_1, z_2, z_3 are the affixes of the vertices of a triangle ABC, discribed in anticlockwise sense,

then $\frac{(z_1 - z_2)}{|z_1 - z_2|} e^{i\alpha} = \frac{(z_1 - z_3)}{|z_1 - z_3|}$ or, amp, $\left(\frac{z_1 - z_3}{z_1 - z_2}\right) = \alpha = \angle BAC$



Note, when $\alpha = \frac{\pi}{2}$, or, $\frac{-\pi}{2}$, then $\frac{z_1 - z_3}{z_1 - z_2}$ is pure imaginary.

De Moiver's Theorem, Proof of De Moiver's theorm :

Statement :

(i) If $n \in Z$ (the set of integers), then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) If $n \in Q$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof :

(i) When $n \in Z$, we know that $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n \Rightarrow e^{i(n\theta)} = (\cos\theta + i\sin\theta)^n \quad \Rightarrow \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$

(ii) Let n be rational number. Let $n=\frac{p}{q}$, where p, q are intergers and $q\neq 0$. From part (i), we have

$$\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)^q=\cos\left(\left(\frac{p\theta}{q}\right)q\right)+i\sin\left(\left(\frac{p\theta}{q}\right)q\right)=\cos p\theta+i\sin p\theta$$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos p\theta+i\sin p\theta)^{\frac{1}{q}}$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $\left[(\cos\theta+i\sin\theta)^p\right]^{\frac{1}{q}}$

$\Rightarrow \cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$

Note

1. De Moiver's theorem is also true for $(\cos\theta-i\sin\theta)$, i.e.,

$(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$, because $(\cos\theta-i\sin\theta)^n=[\cos(-\theta)+i\sin(-\theta)]^n$

$=\cos(-n\theta)+i\sin(-n\theta)=\cos n\theta-i\sin n\theta.$

2. $\frac{1}{\cos\theta+i\sin\theta}=(\cos\theta+i\sin\theta)^{-1}=\cos\theta-i\sin\theta$ **3.** $(\sin\theta\pm i\cos\theta)^n\neq\sin n\theta\pm i\cos n\theta.$

4. $(\sin\theta+i\cos\theta)^n=\left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$

5. $(\cos\theta+i\sin\phi)^n\neq\cos n\theta+i\sin n\phi$

If n is an integer, then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

Note (i) If n any rational number, then $\cos n\theta+i\sin n\theta$ is one of the value of $(\cos\theta+i\sin\theta)^n$.

(ii) $(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta$ (iii) $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

(iv) $\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta$ (v) $(\sin\theta+i\cos\theta)^n=\sin n\theta+i\cos n\theta$

In fact, $(\sin\theta+i\cos\theta)^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$

(vi) $(\cos\theta+i\sin\phi)\neq(\cos n\theta+i\sin n\phi)$

(vii) $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)......(\cos\theta_n+i\sin\theta_n)$

$=\cos(\theta_1+\theta_2+.....+\theta_n)+i\sin(\theta_1+\theta_2+....+\theta_n)$

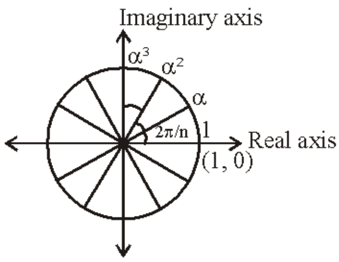
Roots of a Complex Number :

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then $z^{\frac{1}{2}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$

where $k = 0, 1, 2, 3, \ldots (n-1)$

n[th] roots of unity and its geometrical significance :

If $1, \alpha_1, \alpha_2, \alpha_3 \cdots\cdots \alpha_{n-1}$ are the n, n^{th} roots of unity, then :



(a) They are in G.P. with common ratio $e^{i\left(\frac{2\pi}{n}\right)}$

(b) $1^p + \alpha_1^p + \alpha_2^p + \cdots\cdots + \alpha_{n-1}^p = 0$ if p is not an integral multiple on $n = n$ if p is an integral multiple of n.

(c) $(1-\alpha_1)(1-\alpha_2)\cdots\cdots(1-\alpha_{n-1}) = n$ and $(1+\alpha_1)(1+\alpha_2)\cdots\cdots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(d) $1\cdot\alpha_1\cdot\alpha_2\cdot\alpha_3\cdots\cdots\alpha_{n-1} = 1$ or, -1 according as n is odd or even.

Since $1 = \cos 0 + i\sin 0$

$\therefore (1)^{1.n} = (\cos 0 + i\sin 0)^{\frac{1}{n}} = \cos\frac{2\pi r + 0}{n} + i\sin\frac{2\pi r + 0}{n}$ where $r = 0, 1, 2, \ldots..(n-1)$

$= \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n}$ where $r = 0, 1, 2, \ldots. (n-1)$

$= e'\frac{2\pi r}{n}$, where $r = 0, 1, 2, \ldots (n-1)$

$= 1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, e^{i\frac{2(n-1)\pi}{n}} = 1, \alpha, \alpha^2, \ldots..\alpha^{n-1}$ where $\alpha = e^{\frac{i2\pi}{n}}$

Properties of nth Roots of Unity :

(i) $1 + \alpha + \alpha^2 + \cdots\cdots + \alpha^{n-1} = 0$ (ii) $1.\alpha.\,\alpha^2\cdots\cdots\alpha^{n-1} = (-1)^{n-1}$

(iii) The n, nth roots of unity lie on the unit circle $|z| = 1$ and form the vertices of a regular polygon of n sides.

(iv) nth roots of untiy form a G.P. with common ratio $e^{\frac{i2\pi}{n}}$

Relation between nth roots of unity and problems bases on it :

Qu. If $1, \omega, \omega^2, \cdots\cdots, \omega^{n-1}$ are nth roots of unity, find the value of $(5-\omega)(5-\omega)^2\cdots\cdots\left(5-\omega^{n-1}\right)$.

Sol. Let $x = (1)^{\frac{1}{n}}$

$\therefore x^n - 1 = 0$ (has n roots i.e.., $1, \omega, \omega^2, \cdots\cdots, \omega^{n-1}$)

$\Rightarrow \therefore x^n - 1 = (x-1)(x-\omega)\left(x-\omega^2\right)\cdots\cdots\left(x-\omega^{n-1}\right)$

$$\Rightarrow \frac{x^n-1}{x-2}=(x-\omega)\left(x-\omega^2\right)\cdots\cdots\left(x-\omega^{n-1}\right)$$

$\Rightarrow$ Putting $x=5$ in both sides, we get $(5-\omega)\left(5-\omega^2\right)\cdots\cdots\left(5-\omega^{n-1}\right)=\dfrac{5^n-1}{4}$.

- **Cube roots of unity and its geometrical significance :**

Let $z=\sqrt[3]{1} \Rightarrow z^3-1=0 \Rightarrow (z-1)\left(z^2+z+1\right)=0$

$\Rightarrow$either $z=1$ or, $z^2+z+1=0$

$z=\dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$ $\qquad\qquad \therefore\ z=1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

where $\omega=\dfrac{-1+i\sqrt{3}}{2}$, and $\omega^2=\dfrac{-1-i\sqrt{3}}{2}$ $\therefore$ Cube roots of unity are $1, \omega, \omega^2$.

Properties of Cube Roots of Unity

(i) $1+\omega+\omega^2=0$ $\qquad$ (ii) $\omega^3=1$

(iii) $\omega^{3n}=1,\ \omega^{3n+1}=\omega,\ \omega^{3n+2}=\omega^2$ $\quad$ (iv) $\overline{\omega}=\omega^2$ and $(\overline{\omega})^2=\omega,\ \omega\overline{\omega}=\omega^3,\ \omega=e^{i\frac{2\pi}{3}},\ \omega^2=e^{\frac{-i2\pi}{3}}$

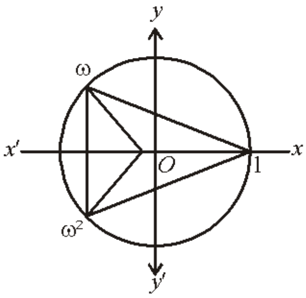
(v) The cube roots of unity lie on the unit circle and divide the circumference into three equal parts.

(vi) $a+b\omega+c\omega^2=0 \Rightarrow a=b=c$ if a, b, c are real.

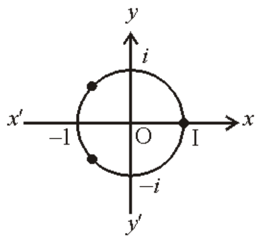
If $1, \omega, \omega^2$ be cube roots of unity and n is a positive integer, then

$$1+\omega^n+\omega^{2n}=\begin{cases}3, & \text{when } n \text{ is a multiple of } 3\\ 0, & \text{when } n \text{ is not a multiple of } 3\end{cases}$$

Cube roots of unity represent vertices of equilateral triangle on the Argand plane.



- ***Some Useful Relations*** :



(i) $x^2+y^2=(x+iy)(x-iy)$

(ii) $x^3+y^3=(x+y)(x+y\omega)\left(x+y\omega^2\right)$

(iii) $x^3-y^3=(x-y)(x-y\omega)\left(x-y\omega^2\right)$

(iv) $x^2+xy+y^2=(x-y\omega)\left(x-y\omega^2\right)$, In particular, $x^2+x+1=(x-\omega)\left(x-\omega^2\right)$

(v) $x^2-x+y^2=(x+y\omega)\left(x+y\omega^2\right)$, In particular, $x^2-x+1=(x+\omega)\left(x+\omega^2\right)$

(vi) $x^2+y^2+z^2-xy-xz-yz=\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right)$

(vii) $x^3+y^3+z^3-3xyz=(x+y+z)\left(x+y\omega+\omega^2 z\right)\left(x+\omega^2 y+\omega z\right)$

Problems on 1, ω, ω^2 and $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$:

Qu. If ω be an imaginary cube root of unity, prove that, $\frac{1}{1+2\omega} - \frac{1}{1+\omega} + \frac{1}{2+\omega} = 0$

Sol. $\frac{1}{1+2\omega} - \frac{1}{1+\omega} + \frac{1}{2+\omega} = \frac{1}{(1+\omega)+\omega} - \frac{1}{1+\omega} + \frac{1}{1+(1+\omega)}$

$$= \frac{1}{\omega - \omega^2} + \frac{1}{\omega^2} + \frac{1}{1-\omega^2} = \frac{\omega(1+\omega) - \omega^2 + 1 + \omega^2}{\omega^2(1-\omega^2)} = \frac{1+\omega+\omega^2}{\omega^2(1-\omega^2)} = 0 \qquad [\because 1+\omega+\omega^2 = 0]$$

Qu. $(x + y\omega + z\omega^2)^2 + (x\omega + y\omega^2 + z)^2 + (x\omega^2 + y + z\omega)^2 = 0.$

Sol. $(x + y\omega + z\omega^2)^2 + (x\omega + y\omega^2 + z\omega^3)^2 + (x\omega^2 + y\omega^3 + z\omega^4)^2$

$$= (x + y\omega + z\omega^2)^2 + \omega^2(x + y\omega + z\omega^2)^2 + \omega^4(x + y\omega + z\omega^2)^2$$

$$= (x + y\omega + z\omega^2)^2(1 + \omega^2 + \omega) \qquad [\because \omega^4 = \omega]$$

$$= (x + y\omega + z\omega^2)^2 \times 0 \qquad [\because 1 + \omega + \omega^2 = 0]$$

$$= 0$$

Qu. The value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$ is

(a) 2 (b) –2 (c) 1 (d) –1

Sol. **(b)** Given expression $= \frac{i^{10}(i^{582} + i^{580} + i^{578} + i^{576} + i^{574})}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$

$$= i^{10} - 1 = (i^2)^5 - 1 = (-1)^5 - 1 \qquad = -1 - 1 = -2.$$

Short Cut Technique

- Complex number as an ordered pair : A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a, b). For a complex number to be uniquely specified, we need two real numbers in particular order.
- $0 = 0 + i.0$, is the identity element for addition.
- $1 = 1 + i0$ is the identity element for multiplication.
- The additive inverse of a complex number $z = a + ib$ is $-z$ (*i.e.* $-a - ib$).
- For every non-zero complex number z, the multiplicative inverse of z is $\frac{1}{z}$.
- $|z| \geq |\text{Re}(z)| \geq \text{Re}(z)$ and $|z| \geq |\text{Im}(z)| \geq \text{Im}(z)$
- $\frac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$
- $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$
- If $\left|z + \frac{1}{z}\right| = a$, the greatest and least values of $|z|$ are respectively $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$.
- $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_2 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$
- If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ or $arg\ z_1 = arg\ z_2$
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg\ (z_1) = arg\ (z_2)$ i.e., z_1 and z_2 are parallel.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg\ (z_1) - arg\ (z_2) = 2n\pi$, where n is some integer.
- $|z_1 - z_2| = ||z_1| - |z_2|| \Leftrightarrow arg(z_1) - arg(z_2) = 2n\pi$ where n is some integer.
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg\ (z_1) - arg\ (z_2) = \pi/2$.
- If $|z_1| \leq 1, |z_2| \leq 1$ then

  (i) $|z_1 + z_2|^2 \leq (|z_1| - |z_2|)^2 + (arg\ (z_1) - arg\ (z_2))^2$. (ii) $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - (arg(z_1) - arg(z_2))^2$.
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- If $|z_1| = |z_2|$ and $amp\,(z_1) + amp\,(z_2) = 0$, then z_1 and z_2 are conjugate complex numbers of each other.

- Multiplication of *i*: Since $z = r(\cos\theta + i\sin\theta)$ and $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ then $iz = \left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$

  Hence, multiplication of *z* with *i* then vector for *z* rotates a right angle in the positive sense. *i.e.*, to multiply a vector by – 1 is to turn it through two right angles. *i.e.*, to multiply a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

- If z_1 and z_2 are two complex numbers then $|z_1 z_2| = r_1 r_2$; $\arg(z_1 z_2) = \theta_1 + \theta_2$ and $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$, *arg* $\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$ and where $|z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

- The area of the triangle whose vertices are *z*, *iz* and *z* + *iz* is $\frac{1}{2}|z|^2$.

- The area of the triangle with vertices z, wz and $z + wz$ is $\frac{\sqrt{3}}{4}|z^2|$.

- If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_o be the circumcentre, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

- If $z_1, z_2, z_3 \ldots\ldots z_n$ be the vertices of a regular polygon of *n* sides and z_0 be its centroid, then $z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = nz_0^2$.

- If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral iff $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

  or $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

- If $z_1, z_2 z_3$ are the vertices of an isosceles triangle, right angled at z_2 then $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

- If z_1, z_2, z_3 are the vertices of a right-angled isosceles triangle, then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

- If z_1, z_2, z_3 be the affixes of the vertices *A*, *B*, *C* respectively of a triangle *ABC*, then its orthocentre is

  $$\frac{a(\sec A)z_1 + b(\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}.$$

- The equation $|z - z_1|^2 + |z - z_2|^2 = k$ (where *k* is a real number) will represent a circle with centre at $\frac{1}{2}(z_1 + z_2)$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{1}{2}|z_1 - z_2|^2$

- $(\overline{iz}) = -i\bar{z}$, $\text{Re}(iz) = -\text{Im}(z), \text{Im}(iz) = \text{Re}(z)$

- If the complex numbers z_1 and z_2 are such that the sum $z_1 + z_2$ is a real number, then they are not necessarily conjugate complex.

- If z_1 and z_2 are two complex numbers such that the product $z_1 z_2$ is a real number, then they are not necessarily conjugate complex.

- Since $|z|^2 = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$, therefore $\text{Re}(z) \leq |z|, \text{Im}(z) \leq |z|$

Solved Example

1. If $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, then amplitude of z (z – 1) is equal to

a) $\frac{5\pi}{6}$ b) $-\frac{5\pi}{6}$

c) π d) $-\frac{2\pi}{3}$

Solution : $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

Its amplitude $=\frac{\pi}{3}$.

$z-1=\frac{1}{2}+\frac{\sqrt{3}}{2}i-1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$\therefore$ Its amplitude $=\frac{2\pi}{3}$

$\therefore$ Amplitude of

$z(z-1)=\text{Amp of } z+\text{Amp of }(z-1)$

$=\frac{\pi}{3}+\frac{2\pi}{3}=\pi$

Answer : (c)

2. If $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$, then amplitude of $\omega=\sqrt{z^2-1}$ is

a) $\frac{2\pi}{3}$ b) $\frac{\pi}{3}$

c) $-\frac{2\pi}{3}$ d) $-\frac{\pi}{3}$

Solution : $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

$\therefore z^2-1=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^2-1=\frac{3}{4}-\frac{1}{4}-\frac{\sqrt{3}}{2}i-1$

$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$=\cos\left(\frac{2\pi}{3}\right)-i\sin\left(\frac{2\pi}{3}\right)=\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)$

$\therefore \text{Amp}\left(z^2-1\right)^{1/2}=\text{Amp}\left\{\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right\}^{1/2}$

$=\text{Amp}\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}=-\frac{\pi}{3}$

Answer : (d)

3. If $z=\frac{1+i}{2}$, then the value of $z^{2^{n+1}}$, $(n\ge 2)$ is equal to

a) 2^{-2^n} b) 2^{-n}

c) 2^{-2n} d) $2^{-2^{n-2}}$

Solution : $z^{2^{n+1}}=z^{2^n.2}=\left(z^2\right)^{2^n}=\left\{\left(\frac{1+i}{2}\right)^2\right\}^{2^n}$

$=\left\{\frac{1-1+2i}{4}\right\}^{2^n}=\left(\frac{i}{2}\right)^{2^n}=\frac{(i)^{2^{n-2}.2^2}}{2^{2^n}}$

$=\frac{\left(i^4\right)^{2^{n-2}}}{2^{2^n}}=\frac{1}{2^{2^n}}=2^{-2^n}$

Answer : (a)

4. If x, y are real and $\sqrt[3]{x+iy}=3+2i$, then the value of $\frac{x}{3}+\frac{y}{2}$ is equal to

a) 16 b) 18
c) 20 d) 22

Solution : $x+iy=(3+2i)^3=27-8i+54i-36$

$=-9+46i$

$\therefore x=-9$ and $y=46$

$\therefore \frac{x}{3}+\frac{y}{2}=-3+23=20$

Answer : (c)

5. If $z_1=2+3i$, $z_2=3-i$ and $z_3=1+2i$, then the simplest value of $z_1\,\text{Im}\left(\bar{z}_2\, z_3\right)$ is equal to (Im z is imaginary part of z)

a) 4 + 8i b) 14 – 21i
c) 4 – 8i d) 14 – 21i

Solution :

$\therefore \bar{z}_2 z_3 = (3+i)(1+2i) = 3-2+7i = 1+7i$

$\therefore z_1\ \mathrm{Im}(\bar{z}_2 z_3) = (2+3i)\times 7 = 14+21i$

Answer : (d)

6. If $z = \dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, then the value of θ $\left(0 \le \theta \le \dfrac{\pi}{2}\right)$ is

a) $\dfrac{\pi}{2}$ b) $\dfrac{\pi}{3}$

c) $\dfrac{\pi}{4}$ d) $\dfrac{\pi}{6}$

Solution

$$z = \frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta}$$

$$= \frac{3-4\sin^2\theta + i(6\sin\theta + 2\sin\theta)}{1+4\sin^2\theta}$$

As z is purely imaginary, its real part is zero.

$\therefore 3-4\sin^2\theta = 0 \Rightarrow \sin\theta = \pm\dfrac{\sqrt{3}}{2}$

$\therefore \theta = \dfrac{\pi}{3}\ \left(0\le\theta\le\dfrac{\pi}{2}\right)$

Answer : (b)

7. If $|z+4| \le 3$, then the maximum value of $|z+1|$ is

a) 6 b) 8

c) 10 d) 5

Solution : $|z+1| = |z+4-3| \le |z+4| + |3|$

$\le 3+3 = 6$

$\therefore$ Required maximum value is 6

Answer : (a)

8. The value of $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is

a) 1 b) –1

c) i d) –i

Solution : $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$

$$= i\sum_{k=1}^{10}\left(\frac{1}{i}\sin\frac{2k\pi}{11} + \cos\frac{2k\pi}{11}\right)$$

$$= i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$$

$$= i\sum_{k=1}^{10}\left\{\cos\left(-\frac{2k\pi}{11}\right) + i\sin\left(-\frac{2k\pi}{11}\right)\right\}$$

$$= i\sum_{k=1}^{10} e^{-ik\theta} \quad \text{where } \theta = \frac{2\pi}{11}$$

$$= i\left[e^{-i\theta} + e^{-2i\theta} + \ldots\ldots + e^{-10i\theta}\right]$$

$$= i e^{-i\theta}\left[1 + e^{-i\theta} + \ldots\ldots + e^{-9i\theta}\right]$$

$$= i e^{-i\theta}\frac{e^{-10i\theta}-1}{e^{-i\theta}-1} = i\frac{\left[e^{-11i\theta} - e^{-i\theta}\right]}{e^{-i\theta}-1}$$

Now, $e^{-i11\theta} = \cos\left(11.\dfrac{2\pi}{11}\right) - i\sin\left(\dfrac{11\times 2\pi}{11}\right)$

$= \cos 2\pi - i\sin 2\pi = 1$

$\therefore$ Required sum $= i\dfrac{\left[1-e^{-i\theta}\right]}{e^{-i\theta}-1} = -i$

Answer : (d)

9. The value of $\left(i+\sqrt{3}\right)^{80} + \left(i-\sqrt{3}\right)^{80} + 2^{80}$ is equal to

a) 2^{80} b) 0

c) 2^{80} d) ω^2

Solution : $\left(i+\sqrt{3}\right)^{80} = i^{80}\left(1-i\sqrt{3}\right)^{80}$

$$= 2^{80} i^{80}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{80} = 2^{80}(-\omega)^{80} = 2^{80}\omega^2$$

Similarly, $\left(i-\sqrt{3}\right)^{80} = 2^{80} i^{80}\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^{80}$

$$= 2^{80}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{80} = 2^{80}\left(\omega^2\right)^{80} = 2^{80}\omega$$

$\therefore$ Required value $= 2^{80}\left(\omega^2 + \omega + 1\right) = 0$

Answer : (b)

10. If z is a complex number such that $|z|=1$, then $\frac{1+z}{1+\bar{z}}$ is equal to

a) $1+\bar{z}$ b) $1+iz$
c) $\bar{z}$ d) z

Solution : For any complex number z, $z\bar{z}=|z|^2$ The given expression

$$=\frac{1+z}{1+\bar{z}}=\frac{z(1+z)}{z+z\bar{z}}=\frac{z(1+z)}{z+1}=z$$

Answer : (d)

11. If $|z-8|=|z-11|$, then the complex number $z(=x+iy)$ lies on

a) circle b) ellipse
c) straight line d) hyperbola

Solution : $|z-8=|z-11||$

$|x-8+iy|=|x-11+iy|$

or, $(x-8)^2+y^2=(x-11)^2+y^2$

or, $x^2-16x+64+y^2=x^2-22x+121+y^2$

or, $6x=57$, which is a straight line.

Answer : (c)

12. If $z=x+iy$ and $|z-1|^2-(z-1)^2=2$, then x and y are are equal to

a) 0, 0 b) $1, \pm 1$
c) 0, 2 d) 1, 2

Solution : $|z-1|^2-(z-1)^2=2$

or, $|x-1+iy|^2-(x-1+iy)^2=2$

or, $(x-1)^2+y^2-\{(x-1)^2-y^2+2i(x-1)y\}=2$

or, $2y^2-2i(x-1)y=2$

or, $2y^2=2\Rightarrow y=\pm 1$

and $2(x-1)y=0 \Rightarrow x=1 \quad (\because y=\pm 1)$

$\therefore x=1,\ y=\pm 1$

Answer : (b)

13. If z is a complex number and $|z+3+4i|\le 8$, then maximum value of $|z+9+12i|$ is

a) 10 b) 14
c) 16 d) 18

Solution : $|z+9+12i|=|(z+3+4i)+(6+8i)|$

$\le |z+3+4i|+|6+8i|$

$\le 8+\sqrt{36+64}=8+10=18$

Answer : (d)

14. If ω is cube root of unity and $\omega^{259}+\omega^{259n}=-1$, then n is equal to

a) 3k + 2 b) 3k + 1
c) 3k d) none of these

where k = 0, 1, 2, ...

Solution : $\omega^{259}+\omega^{259n}=-1$

or, $\omega^{3\times 86+1}+\omega^{(3\times 86+1)n}=-1$

or, $(\omega^3)^{86}\omega+(\omega^3)^{86n}.\omega^n=-1$

or, $\omega+\omega^n=-1$

$\therefore$ n = 2, 5, 8 ..... $(\because \omega+\omega^2=-1)$

$\therefore$ n = 3k + 2, k = 0, 1, 2

Answer : (a)

15. If $|z-2+3i|=0$, where z = x + iy, then x and y are respectively equal to

a) 2, –3 b) –2, 3
c) –2, –3 d) 2, 3

Solution : $|x-2+(y+3)i|=0$

or, $(x-2)^2+(y+3)^2=0$

$\therefore (x-2)^2+(y+3)=0 \Rightarrow x=2\ \&\ y=-3$

Answer : (a)

16. If $\omega^3=1$ and ω^2 lies in 3rd quadrant of Argand plane, then $Arg(\omega)$ is equal to

a) $-\frac{\pi}{3}$ b) $\frac{2\pi}{3}$

c) $-\frac{2\pi}{3}$ d) $\frac{\pi}{3}$

Solution : Here $\omega^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and

$\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

i.e. $\omega=\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right) \quad \therefore Arg\omega=\frac{2\pi}{3}$

Answer : (b)

17. If $z = \cos\theta + i\sin\theta$, then $\cos 2\theta$ is equal to

a) $\dfrac{z+\bar{z}}{2}$ b) $\dfrac{z^2-\bar{z}^2}{2}$

c) $\dfrac{z^2-\bar{z}}{2}$ d) $\dfrac{z^2+\bar{z}^2}{2}$

Solution :

$z^2 = (\cos\theta + i\sin\theta)^2 = \cos^2 - \sin^2\theta + 2i\sin\theta\cos\theta$

$= \cos 2\theta + i \sin 2\theta$

$\bar{z}^2 = (\cos\theta - i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta - 2i\sin\theta\cos\theta$

$= \cos 2\theta - i \sin 2\theta$

$\therefore z^2 + \bar{z}^2 = 2\cos 2\theta$

$\therefore \cos 2\theta = \dfrac{z^2+\bar{z}^2}{2}$

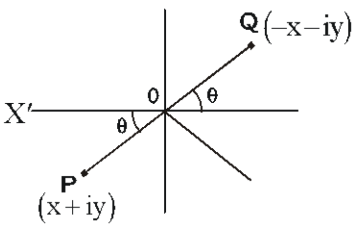
Answer : (d)

18. The point $z = x + iy$ lies in the 3rd quadrant of the Argand plane, the argument of $\omega = \dfrac{x+iy}{-x-iy}$ is

a) $-\pi$ b) 0

c) π d) $\dfrac{\pi}{2}$

Solution :



As $x+iy$ lies in 3rd quadrant, x and y are both –ve. Hence $-x-iy$ lies in first quadrant. Let $\angle X'OP = \theta$

$\text{Arg}\dfrac{x+iy}{-x-iy} = \text{Arg}(x+iy) - \text{Arg}(-x-iy)$

$= (-\pi+\theta) - (\theta) = -\pi$ But as $-\pi < \text{Arg} \le \pi$,

required argument = π

Answer : (c)

19. If ω is a complex cube root of unity, then the value of

$$\cos\left[\left\{(1-\omega)(1-\omega^2)+(1-2\omega)(1-2\omega^2)+\ldots+(1-10\omega)(1-10\omega^2)\right\}\frac{\pi}{900}\right]$$

is equal to

a) 0 b) $\dfrac{1}{2}$

c) $-\dfrac{1}{2}$ d) $\dfrac{\sqrt{3}}{2}$

20. If $z_k = \cos\dfrac{k\pi}{10} + i\sin\dfrac{k\pi}{10}$, Then $z_1 z_2 z_3 z_4$ is equal to

a) 0 b) $1+i$

c) $1-i$ d) -1

Solution :

$z_1 z_2 z_3 z_4 = \left(\cos\dfrac{\pi}{10}+i\sin\dfrac{\pi}{10}\right)\left(\cos\dfrac{2\pi}{10}+i\sin\dfrac{2\pi}{10}\right)\times$

$\left(\cos\dfrac{3\pi}{10}+i\sin\dfrac{3\pi}{10}\right)\left(\cos\dfrac{4\pi}{10}+i\sin\dfrac{4\pi}{10}\right)$

$= \cos\left(\dfrac{\pi}{10}+\dfrac{2\pi}{10}+\dfrac{3\pi}{10}+\dfrac{4\pi}{10}\right)+i\sin\left(\dfrac{\pi}{10}+\dfrac{2\pi}{10}+\dfrac{3\pi}{10}+\dfrac{4\pi}{10}\right)$

$= \cos\pi - i\sin\pi = -1$

Answer : (d)

21. The argument and modulus of $\left(\sqrt{3}+1\right)^6$ is

a) $\pi, 2^3$ b) $\pi, 2^6$

c) $\dfrac{\pi}{2}, 2^8$ d) $\dfrac{\pi}{2}, 2^3$

Solution : $\left(\sqrt{3}+i\right)^6 = 2^6\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)^6$

$= 2^6\left\{\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right\}^6 = 2^6(\cos\pi + i\sin\pi)$

$\therefore$ Argument = π and modulus = 2^6

Answer : (b)

22. The modulus of $\dfrac{(3+4i)(1-i)(6+8i)}{4-3i}$ is

a) $10\sqrt{2}$ b) $10\sqrt{3}$
c) $10\sqrt{4}$ d) none

Solution : $|3+4i| = \sqrt{3^2+4^2} = 5$

$|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$

$|6+8i| = \sqrt{6^2+8^2} = 10$

$|4-3i| = \sqrt{16+9} = 5$

$\therefore$ Required modulus $= \dfrac{5\times\sqrt{2}\times 10}{5} = 10\sqrt{2}$

Answer : (a)

23. The imaginary part of $\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^{80}$ is

a) $\dfrac{1}{2}$ b) $-\dfrac{1}{2}$

c) $-\dfrac{\sqrt{3}}{2}$ d) $\dfrac{\sqrt{3}}{2}$

Solution : $= \left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^{80} = (-1)^{80}\left\{-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right\}^{80}$

$= \omega^{80} = \omega^{3\times 26+2} = \omega^2 = -\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

$\therefore$ The required imaginary part $= -\dfrac{\sqrt{3}}{2}$

Answer : (c)

24. The amplitude of $\left(\sin\dfrac{\pi}{12}+i\cos\dfrac{\pi}{12}\right)^{8}$ is

a) $\dfrac{2\pi}{3}$ b) $-\dfrac{2\pi}{3}$

c) $-\dfrac{5\pi}{6}$ d) $\dfrac{5\pi}{6}$

Solution :

$\left(\sin\dfrac{\pi}{12}+i\cos\dfrac{\pi}{12}\right)^{8} = i^8\left(\cos\dfrac{\pi}{12}+\dfrac{1}{i}\sin\dfrac{\pi}{12}\right)^{8}$

$= \left(\cos\dfrac{\pi}{12}-i\sin\dfrac{\pi}{12}\right)^{8} = \left\{\cos\left(-\dfrac{\pi}{12}\right)+i\sin\left(-\dfrac{\pi}{12}\right)\right\}^{8}$

$= \cos\left(-\dfrac{8\pi}{12}\right)+i\sin\left(-\dfrac{8\pi}{12}\right)$

$= \cos\left(-\dfrac{2\pi}{3}\right)+i\sin\left(-\dfrac{2\pi}{3}\right)$

$\therefore$ Required argument $= -\dfrac{2\pi}{3}$

Answer : (b)

25. If ω is a imginary cube root of unity and α, β are roots of $x^2-2x+2=0$, then the value of $(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)(\omega^2\alpha+\omega^2\beta)$ is equal to

a) 2 b) $-4\omega^2$
c) 4 d) -2

Solution : $\alpha+\beta=2$ and $\alpha\beta=2$

$(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)(\omega^2\alpha+\omega^2\beta)$

$= \omega^4(\alpha+\omega\beta)(\omega\alpha+\beta)(\alpha+\beta)$

$= 2\omega(\alpha+\omega\beta)(\omega\alpha+\beta)$

$= 2\omega\left\{\alpha^2\omega+\beta^2\omega+\alpha\beta(1+\omega^2)\right\}$

$= 2\omega\left[\omega\left\{(\alpha+\beta)^2-2\alpha\beta\right\}+\alpha\beta(1+\omega^2)\right]$

$= 2\omega\left[\omega(4-4)+2(-\omega)\right] = -4\omega^2$

Answer : (b)

26. If $(x+iy)^{1/3} = 4+2i$, then the value of $\dfrac{x}{2}+\dfrac{y}{11}$ is

a) 13 b) 9
c) 8 d) 16

Solution : $(x+iy) = (4+2i)^3 = 2^3(2+i)^3$

$= 8(8+12i-6-i) = 8(2+11i)$

$\therefore x = 16$ and $y = 88$

$\therefore \dfrac{x}{2}+\dfrac{y}{11} = 8+8 = 16$

Answer : (d)

27. If $i=\sqrt{-1}$ and ω is an imaginary cube root of unity, then the value of $(i\omega)(i\omega^2)(i\omega^3)(i\omega^4)...$ upto 101 terms is equal to

a) -1 b) $-\omega$

c) i d) $-\omega^2$

Solution : Required value $=(i\omega)(i\omega^2)(i\omega^3)(i\omega^4)...$ upto 101 terms —

$$= i^{101}\times\omega^{1+2+3+...+101} = i^{4\times25+1}\omega^{\frac{101\times102}{2}}$$

$$= i\times\omega^{101\times51} = i\left(\omega^{3\times17\times101}\right) = i\left(\omega^3\right)^{17\times101} = i$$

Answer : (c)

28. If $16z^4 - i = 0$, then z can assume the value of

a) $\frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

b) $\frac{1}{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$ c) $2i$

d) $\frac{1}{2}(1+i)$

Solution : $(2z)^4 = i = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$$\therefore 2z = \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{\frac{1}{4}}$$

$\therefore \cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$ is one of the values of 2z.

i.e. $\frac{1}{2}\left\{\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right\}$ is one of the values of z.

Answer : (b)

29. The complex conjugate of

$$z=\frac{(2+i)(1+2i)i}{(1+i)^2}$$ is

a) $-\frac{5}{2}i$ b) $1-3i$

c) $1+3i$ d) $3+i$

Solution :

$$z=\frac{(2+i)(1+2i)i}{(1+i)^2}=\frac{(2-2+5i)i}{1-1+2i}=\frac{5i^2}{2i}=\frac{5}{2}i$$

$\therefore$ Required conjugate $= -\frac{5}{2}i$

Answer : (a)

30. If $(1+i)(2+i)(3+i)....(10+i)=A+iB$, then the value of $(1^2+1)(2^2+1)(3^2+1)...(10^2+1)$ is equal to

a) A^2-B^2 b) $(A+B)^2$

c) $(A-B)^2$ d) A^2+B^2

Solution :

$(1+i)(2+i)(3+i)....(10+i)=A+iB$ ....(1)

$\therefore (1-i)(2-i)(3-i)....(10-i)=A-iB$ ....(2)

Multiplying both sides of (1) and (2)

$(1^2+1)(2^2+1)...(10^2+1)=A^2+B^2$

Answer : (d)